"OPTIMAL-RANGE-VELOCITY-POLAR"

A NEW THEORETICAL TOOL FOR THE OPTIMIZATION OF SAILPLANE FLIGHT TRAJECTORIES

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SUMMARY

On a cross-country flight a sailplane pilot may optimize his average cross-country speed by adjusting his instantaneous horizontal velocity (and thereby his instantaneous vertical velocity) so that he flies faster thru regions with downward moving air and slower thru regions with upward moving air. For the exact solution of this optimization problem in case of a given arbitrary vertical atmospheric velocity distribution along the course, a simple new tool is introduced in this paper in the form of the definition of an "optimal-range-velocity-polar" or, ORV-polar. This ORV-polar is the plot which provides the optimal average vertical velocity of the sailplane over the course as a function of its average horizontal velocity.The shape, the properties, the construction and the use of the ORV-polar are discussed in this paper. In particular it is shown that the optimal velocity histories which correspond to the individual points of the ORV-polar are each dependent on only one quantity, the so called "McCready-ring setting." As a result, these optimal velocity histories may be generated in practice in a relatively easy way with aids and/or instruments currently in use by the sailplane pilots.

For theoretical purposes the ORV-polar concept facilitates the understanding of known theoretical results, such as the rule that (ignoring the possibility of an early landing by lack of height) the optimal velocity history over the total range is completely determined by the largest possible net rate of climb encountered along the course. Also, the concept of the ORV-polar makes it easy to understand that flying S-curves, as proposed by some authors, when optimal, is never the only optimal strategy.

For practical purposes the ORV-concept makes it feasible to determine the exact optimal McCready-ring-setting for any course with any vertical atmospheric velocity distribution. For the special case of a square-wave thermal model, the optimal McCready-ring-setting may be determined by a simple graphical method which requires no more information than the velocity polar (i.e., the regular relationship between the horizontal and vertical velocity) of the sailplane. As such, this particular optimal McCready-ring-setting can be determined by any sailplane pilot without the aid of a computer. As an example of this last use of the ORV-polar concept, the paper also presents the optimal McCready-ring-settings for a variety of square-wave-thermal-model values for a particular sailplane type (LS-3) representative of modern racing-class sailplanes.

SYMBOLS

e ($=L_2/L$)	(cloud street) extension factor
L	length of range
T	time of travel over range
u	vertical atmospheric velocity
v	horizontal velocity of sailplane
w	vertical velocity of sailplane
z	Lagrange multiplier value or McCready-ring-setting (or net rate of climb)
λ	Lagrange multiplier

SUBSCRIPTS

$i,1,2,\ldots$	relate to the 1th,1st,2nd,..part of range
a	relates to the atmosphere
av	relates to the average value
max	relates to the maximum value

mind	relates to minimum rate of descent
mr	relates to minimal value over the range
msf	relates to MSF (=minimal-straight-flight-) point
opt	relates to solution of an optimization problem
orv	relates to ORV (=optimal-range-velocity-) vector
p	relates to velocity polar, i.e., to the sailplane relative to the surrounding air
s	relates to synthesis of two or more ORV-polars
th	relates to the thermal
xl	relates to the ZL (=zero-(altitude-) loss) point

NOTATIONAL AIDS

—	relates to use of the extended velocity polar
^	relates to the solution of the optimization problem
~	proportional to

1. INTRODUCTION

A sailplane may travel over great distances when the pilot gains altitude in regions of rising air and subsequently transforms this altitude into distance by gliding thru regions of sinking or still air. For a given sailplane in equilibrium flight, there exists a (usually known) relationship between the horizontal velocity of the plane and its vertical velocity relative to the air and this provides the pilot with the option of trading altitude loss for speed over the descent part of his trajectory. The determination of the best speed to fly to optimize the average velocity along the course, taking into account the time spent in gaining altitude, is an interesting optimization problem that has been attacked by a number of theory-minded sailplane pilots and optimization specialists over the years.

In the earliest formulation of the problem of the cross-country flight of sailplanes, the case considered was that of altitude gained exclusively in small local regions (thermals) with relatively strong vertical atmospheric velocity with given fixed magnitude and that gliding takes place through a region of still air (Fig. 1). The problem in this case consists of the determination of the (constant) cruise velocity between thermals which results in the shortest time to fly from a point (Pt. A, Fig. 1) in one thermal to a point (Pt. C, Fig. 1) at the same height in the following thermal. The solution to this algebraic optimization problem, already knownto some German competition sailplane pilots before WWII, became common knowledge to the sailplane community after the success in 1948 of the American WorldChampionship pilot Paul McCready who invented a simple device, the McCready-ring, to implement the optimal solution in actual practice. Since then the problem formulation is usually referred to as the McCready problem.

In practice the atmosphere between two thermals will seldom be completely at rest and quite often there will be some vertical atmospheric velocity distribution along the course, As long as this vertical atmospheric velocity is constant over parts of the total course a simple extension of the McCready theory provides the optimal strategy directly. In case of a varying distribution, the determination of the best instantaneous cruise velocity becomes a (simple) problem in the realm of the calculus of variations, the solution of which can be easily derived[2]. The implementation of this solution may be realized in practice quite simply with the earlier mentioned McCready-ring or its recently developed mechanized version, the so-called "Sollfahrtgeber" or speed director[10].

The character of the optimal solution in case of a varying vertical atmospheric velocity distribution is in general such that one should fly faster the stronger the downwards atmospheric velocity and slower the stronger the upward atmospheric velocity. The trajectory of a sailplane thus flying at optimal cruise speeds resembles the trajectory of a jumping dolphin and this mode of flying of sailplanes at optimal cruise speeds has therefore become know as "dolphin-soaring"[11].

In a number of situations, for instance in case of flights under cloud formations known as cloud streets, it may happen that in this type of dolphin flight altitude is gained instead of lost, in which case the pilot no longer has to use thermals to gain altitude: he may fly over long distances in straight flight without circling! Especially during the last ten years, this type of dolphin soaring, also made possible by the advent of glass fiber sailplanes with very high performance characteristics, has resulted in a number of record breaking flights. Dolphin flying strategies are practiced frequently over stretches during regular cross-country flights.

The determination of the optimal cruise velocities in cases where there are large enough regions along the course to permit cross country flying without circling has been the subject of a number of studies. In the earliest of these[1,+] heuristic arguments were used to arrive at good or roughly optimal strategies. Later, studies[2,6] formulated the problem as a (simple) problem in the calculus of variations and arrived at the correct mathematical characterization of the optimal solution. These studies also provided rules for the computation of the optimal solution in any given situation. Most studies thereafter[7,8] applied the theory to simple periodical vertical atmospheric velocity distributions employing sailplanes with mathematically simple performance characteristics. Only very recently[3] has attention been paid to a nonperiodic vertical atmospheric distribution with some suggestions given towards a possible solution.

In the present paper attention will be paid to the solution of the McCready problem which, together with its implementation in practice with such aids as the McCready-ring or the "Sollfahrtgeber", plays a central role in all sailplane trajectory problems.

This discussion will be presented in Section 2 where, in addition, the general dolphin soaring problem will be defined and its known solution briefly reviewed. In Section 3, a particular concept, believed not to have been used earlier within this theory, the optimal-range-velocity-polar (ORV-polar) will be introduced and some properties of it discussed. These properties will turn out to be such that the ORV-polar, which contains all the information for the complete solution of the dolphin-flying problem, can be evaluated in practice in a relatively simple manner.

In Section 4, the latter aspect will be elaborated. Next, in Section 5 the theory will be applied to the case of a square wave velocity distribution and some numerical results will be presented for a particular sailplane of the racing-class type. Finally, in Section 6 some concluding remarks about the use of the ORV-polar in theory and practice will be summarized. The paper closes with two appendices in which the proof of a mathematical and a geometric property of the ORV-polar are given. Not considered in this paper are the dynamical aspects of sailplane trajectory problems[9]. Problems in which a vertical variation of the vertical atmospheric velocity distribution is assumed or problems in which a realistic lower limit of the feasible flight level are also neglected. All these aspects of the sailplane trajectory problem should, among others, be taken into account before one can say that the deterministic sailplane trajectory optimization problem is fully solved.

2. PROBLEM FORMULATION, SOLUTION AND IMPLEMENTATION

2.1 The McCready Problem

Fundamental to all sailplane trajectory optimization problems is the classical McCready problem which is concerned with the question of how fast a sailplane pilot should fly between isolated thermals of given strength in order to minimize the time to fly from a point A (Fig. 1) in one thermal to a point C at the same height in the next thermal. This time can be split up into the time of flight from point A to the first point B to point C in that thermal. The latter time will be determined by the net rate of climb z_{th} in the thermal which is equal to the sum of the vertical atmospheric velocity u_{th} in the thermal and the vertical velocity w_p of the sailplane in circling flight.

If it is assumed that the vertical velocity when circling is equal to the minimum rate of descent, or equivalently the maximum vertical velocity, $w_{p,max}$, in equilibrium flight, then the rate of climb in the thermal will be given by

$$z_{th} := u_{th} + w_{p,max} \tag{2.1}$$

When the difference between the two thermals is L and the sailplane flies in between the two thermals with a (constant) horizontal velocity v_p and a (constant) vertical velocity w_p, then the time of flight from A to B will be L/v_p and the corresponding altitude loss $-(L/v_p)\cdot w_p$. The total time of flight from point A to C therewith becomes

$$T = \frac{L}{v_p} - \frac{L}{v_p}\frac{w_p}{z_{th}} = \frac{L}{z_{th}}\left(\frac{z_{th} - w_p}{v_p}\right) \tag{2.2}$$

For this expression the assumption is essential that for sailplanes w_p will always be negative.

In case of an equilibrium glide in between the thermals, a fixed aircraft weight, a constant air density and a constant gravitational acceleration, the vertical velocity w_p of the sailplane (relative to the air) will depend on its horizontal velocity (relative to the air) according to some functional relationship which is knownas the velocity polar of the particular sailplane (for given aircraft weight (or equivalently given wing-loading) and given air density).

$$w_p = w_p(v_p) \tag{2.3}$$

A sketch of a typical velocity polar for a sailplane is given in Fig. 2. Note in particular that $w_p(v_p)$ is a concave function with a well defined maximum and that the function $w_p(v_p)$ is not defined for speeds smaller than some minimum speed (i.e, the stall speed).

Taking into account the functional relationship (2.3), the solution, i.e., the optimal value of v_p, of the McCready problem, will be characterized by the necessary condition for a minimum of (2.2) which reads

$$w_p(v_p) - v_p \frac{dw_p}{dv_p}(v_p) = z_{th} \tag{2.4}$$

This relation is often referred to as the McCready-relation.

It may be noted that the distance L between the thermals is not present in this expression implying that in theory the optimal solution is independent of the distance. In practice, of course, the distance L does play a role since this distance appears linearly in the altitude loss $-L\ w_p/v_p$ which should not exceed the original height.

The McCready relation has a simple geometric interpretation which is sketched in Fig. 2. In particular, this interpretation makes it possible to construct the optimal horizontal velocity $\hat{v}_p$ as soon as the net rate of climb z_{th} in the next thermal is known by drawing a line through the point $(0, z_{th})$ tangent to the graph of the velocity polar. Of course, in actual practice, the net rate of climb z_{th} of the next thermal will not be known beforehand and therefore use will have to be made of an estimated value of this quantity.

In case the atmosphere between the thermals is not at rest but instead has a constant vertical velocity u_a then, of course, the altitude loss from point A to B will no longer be given by $-(L/v_p)w_p$ but instead by $-(L/v_p)(w_p+u_a)$ and the total time of flight (2.2) by

$$T = \frac{L}{z_{th}}\left(\frac{z_{th} - w_p - u_a}{v_p}\right) \tag{2.5}$$

The McCready relation (2.4) changes accordingly into

$$w_p(v_p) - v_p \frac{dw_p}{dv_p}(v_p) = z_{th} - u_a \tag{2.6}$$

Since length L is not present in (2.6), the McCready-relation will also apply to any part of the trajectory where the vertical atmospheric velocity happens to be constant and which therefore may be considered part of a larger trajectory (of length L) with the given vertical atmospheric velocity over the whole trajectory. For the geometric construction of the optimal velocity, one can either choose to draw a line tangent to the graph of the velocity polar starting out from point $(0, z_{th} - u_a)$ or, equivalently, draw a line from the point $(0, z_{th})$ tangent to a velocity polar moved upwards by the amount u_a. The first construction method is obviously to be preferred from a practical point of view; the second may be preferred from a theoretical point of view.

2.2 McCready-ring and Sollfahrtgeber

Given the relatively straightforward characterization (2.6) of the optimal solution, it is not surprising that means have been sought for mechanizing this solution in terms of the quantities that the pilot generally has at his disposal in flight. These quantities are: 1) the sum $(u_a + w_p)$ of the atmospheric descent velocity and the sailplane's own descent velocity, which sum is measured by the variometer (=rate-of-climb indicator); 2) the velocity $v = (v_p^2 + w_p^2)^{\frac{1}{2}}$ relative to the air, which for the usual sailplane flight trajectory is approximately equal to the horizontal velocity v_p; and 3) an educated guess or estimate z_{th} of the net rate-of-climb in the next thermal.

Best known among the devices for determining the optimal cruise velocities in flight is the so-called McCready-ring[10]. This is a movable ring with a matching (linear) scale around the variometer on which ring appropriate values of the horizontal velocity v_p are inscribed at the (negative) scale locations $v_p \frac{dw_p}{dv_p}$ (determined beforehand from the appropriate velocity polar). Accordingly, at the zero point of the scale on the ring the value $v_{p,mind}$ the velocity for minimum descent is inscribed together with some zero pointer. When the ring is turned such that the zero pointer points towards a value z_{th} on the variometer then the inscribed velocity values v_p will be present opposite to scale values of the variometer equal to $z_{th} + v_p \frac{dw_p}{dv_p}$. In flight, the variometer provides the pilot with a reading of the value of the quantity $u_a + w_p(v_p)$. In order to fly optimally for a given estimate z_{th} of the net rate-of-climb in the next thermal, the pilot has to do no more than to set the pointer of the ring on the particular z_{th} value on the variometer and then adjust his speed such that the pointer of the variometer points towards the inscribed value of velocity actually flown. He then will have achieved his actual vertical velocity

$u_a + w_p(v_p)$ indicated by the pointer of the variometer equal to the scale value

$z_{th} + v_p \frac{dw_p}{dv_p}(v_p)$ on the ring.

In actual practice the use of the McCready ring requires the pilot to continuously match the readings of two instruments, the variometer (with ring) and the airspeed indicator. Of course, this is not an ideal situation and a number of devices have been proposed to facilitate the use of the McCready-ring in practice. By far the simplest to use is the recently developed "Sollfahrtgeber"[10] which is essentially a completely new instrument which directly provides a reading for the quantity

$$u_a + w_p(v_p) - v_p \frac{dw_p}{dv_p}(v_p) .$$

With this instrument, the only thing the pilot has to do to fly optimally is adjust his airspeed in such a way that the pointer of the "Sollfahrtgeber" points towards the value z_{th} of the estimated net rate-of-climb in the next thermal.

It may be noted that for both the McCready ring and the "Sollfahrtgeber" the only information the pilot has to supply for the practical implementation of the optimal solution is the value z_{th} of the estimated net rate-of-climb in the next thermal. It will be shown that the same holds for the practical implementation of more general optimal dolphin-flight-strategies for which the pilot has to supply again one characteristic value similar to z_{th} which value appropriately will be called the McCready-ring setting. The determination of the McCready-ring setting in more general situations will take up a large part of the discussions to follow:

2.3 The Dolphin Soaring Problem

In actual practice the McCready-ring and the "Sollfahrtgeber" are used in a continuous fashion, i.e., the pilot adjusts (in case of a varying vertical atmospheric velocity distribution $u_a(x)$, $x \in [0,L]$, his instantaneous horizontal velocity $v_p(x)$ ideally in such a fashion that at any point x the McCready-relation (2.6) will be satisfied

(2.7)

$$w_p(v_p(x)) - v_p(x) \frac{dw_p}{dv_p}(v_p(x)) = z_{th} - u_a(x)$$

Under the assumption that the relationship between the horizontal velocity v_p and the vertical velocity $w_p(v_p)$ given by the velocity polar (2.3) remains valid when these velocities are varying in time, it can be shown[8] that this quasi-static use of the McCready-relation (2.7) will yield the optimal solution as long as at arrival in the next thermal (with the net rate of climb z_{th}) there is some altitude loss which should be taken care of. The proof of this is similar to the proof for the more general problem to be discussed next and is therefore not given here.

There are occasions, such as the case of cloud streets over part of the total trajectory, that the use of the McCready-ring fed with the proper value of net rate-of-climb z_{th} in the next thermal results in an altitude gain instead of altitude loss at arrival at the next thermal. In that case no circling in that thermal is necessary and the pilot might consider flying faster to reduce this altitude gain and increase his average velocity over the range under consideration. The classical McCready theory no longer applies and instead a new problem may be formulated: how to select the instantaneous horizontal velocity $v_p(x)$ in regions of varying rising and sinking air such that the overall average horizontal velocity is maximized while ending up at a given altitude gain (or loss). In mathematical terms this leads to the constrained minimization problem

(2.8)

$$\min \left\{ \int_0^L \frac{dx}{v_p(x)} \;\middle|\; \int_0^L \frac{w_p(v_p(x)) + u_a(x)}{v_p(x)}\, dx = L \tan \gamma \right\}$$

This problem is generally referred to as the pure dolphin soaring problem. It is a special case of the general sailplane trajectory optimization problem which may be stated as (cf. Fig. 3a)

$$(2.9) \qquad \min \left\{ \int_0^L \frac{dx}{v_p(x)} - \frac{\Delta h}{z_{th}} \;\middle|\; \Delta h = \int_0^L \frac{w_p(v_p(x)) + u_a(x)}{v_p(x)}\, dx \le L \tan \gamma \right\}$$

This latter problem formulation (2.9) differs from the former (2.8) only through the assumed presence of an isolated thermal at some point (not necessarily an end point) of the range.

Another way to account for this situation is to assume that circling in a thermal may be replaced, for the sake of modeling, by a climb over an assumed arbitrary small width of the thermal with a corresponding arbitrary small horizontal velocity. With this assumption, the simpler dolphin soaring problem formulation (2.8) may be used to describe the general sailplane trajectory optimization problem which, as such, will be referred to as the generalized dolphin soaring problem.

In the absence of distinct isolated thermals, and given the usual form of the velocity polar (cf. Fig. 2), the pure dolphin soaring problem (2.8) will in general have no solution unless there is an extensive part of the range over which the vertical atmospheric velocity $u_a(x)$ is larger than the minimum sink rate, $w_{p,max}$, of the sailplane, i.e., unless over part of the range (cf. (2.1))

$$(2.10) \qquad z(x) := u_a(x) + w_{p,max} > 0$$

If this inequality is satisfied over a fraction of the range which is too small to allow pure dolphin flight, i.e., to allow a solution of (2.8), then the pilot has still another possibility to avoid circling in the next thermal and that is to fly S-curves in the region where (2.10) is satisfied. The effect of this "S-ing" is that the horizontal velocity of the sailplane in the direction of the course decreases while its vertical velocity remains the same. The option of S-ing as a possible solution to the dolphin soaring problem (2.8) was first considered by Metzger and Hedrick[8] who took into account this "S-ing-mode", as they called it, by defining an extended velocity polar as the graph of the relation (cf. Fig. 2)

$$(2.11) \qquad \bar{w}_p(v_p) := w_{p,max} \qquad \text{if } v_p \leq v_{p,mind}$$

$$:= w_p(v_p) \qquad \text{if } v_p > v_{p,mind}$$

where $w_p(v_p)$ is the regular velocity polar relation (2.3) and v_{pmind} is the horizontal velocity corresponding to $w_{p,max}$ (cf. Fig. 2). The basic idea of the extended velocity polar will play an important role in the discussion to follow.

For a given vertical atmospheric velocity distribution $u_a(x)$ the generalized dolphin soaring problem (2.8) is a simple calculus-of-variations problem[2] with a subsidiary constraint of the isoperimetric type. For the solutions of such a problem use can be made of the Lagrange multiplier technique[5] which, in this particular case, results in the necessary condition (Euler-Lagrange-equation) for the optimal solution $\hat{v}_p(x),\ x \in [0,L]$.

$$\frac{d}{dv_p}\left[\frac{1}{v_p} - \lambda\left(\frac{\bar{w}_p(v_p) + u_a}{v_p}\right)\right]_{(v_p = \hat{v}_p(x))} = 0$$

or, worked out,

$$(2.12)$$

$$\bar{w}_p(\hat{v}_p(x)) - \hat{v}_p(x)\,\frac{d\bar{w}_p}{dv_p}(\hat{v}_p(x)) = 1/\lambda - u_a(x)$$

in which expression $\lambda(\neq 0)$ is the Lagrange multiplier, which is a constant[5] in case of isoperimetric problems. The value λ should be determined from the subsidiary condition

$$(2.13) \qquad \int_0^L \frac{\bar{w}_p(\hat{v}_p(x)) + u_a(x)}{\hat{v}_p(x)}\, dx = L \tan \gamma$$

The equations (2.12) and (2.13) together completely determine the (optimal) solution of the generalized dolphin soaring problem (2.8). For the determination of the unknown value of $1/\lambda$, which, in view of the similarity between (2.7) and (2.12) may be interpreted as a fixed McCready-ring setting for the range under consideration, use may be made of an iterative procedure consisting of guessing a value for $1/\lambda$, evaluating from (2.12) the corresponding values of $v_p(x)$ and from the integral in (2.13) the corresponding altitude gain or loss. Depending on the latter result $1/\lambda$ is thereafter increased in case of an altitude surplus and decreased in case of an altitude deficit.

Although the described iterative procedure usually converges relatively rapidly, the method is still too complicated to determine in practice the optimal McCready-ring setting $1/\lambda$ for any actual vertical atmospheric velocity distribution encountered. Therefore, the optimal McCready-ring setting has only been evaluated for some special vertical atmospheric velocity profiles such as the sinusoidal distribution[2] and the square-wave distribution[7]. The results thus obtained serve as a guide and provide an estimate for the proper McCready-ring setting for the more general situations in practice.

In the following sections a slightly different approach will be shown to yield the same results.

3. THE OPTIMAL-RANGE-VELOCITY POLAR (ORV-POLAR)

3.1 The Concept of the ORV-polar

A good starting point for the discussion of the ORV-polar concept is the simple observation that given any range (0,L) with any vertical atmospheric velocity distribution $u_a(x)$, $x \in (0,L)$, there will in general be an infinite number of horizontal velocity histories $v_p(x)$, $x \varepsilon (o,L)$ which yield the same average (horizontal) velocity v_{av} over the range under consideration. This observation will be true for arbitrary average velocities $v_{av} > 0$ if one allows circling or "S-ing" (cf Section 2.3) in certain regions of the range. Of the velocity histories which yield a particular average velocity v_{av} the one (or the ones) of most interest for optimization purposes is that which result(s) in the smallest altitude loss or largest altitude gain over the range, or which result(s) in the largest average vertical velocity (= smallest average descent velocity) over the range in question, i.e., the solution of optimization problem

(3.1)

$$\max \left\{ \frac{v_{av}}{L} \int_0^L \frac{w_p(v_p(x)) - u_a(x)}{v_p(x)} dx \;\middle|\; \frac{v_{av}}{L} \int_0^L \frac{dx}{v_p(x)} = 1 \right\}$$

This problem is of the same type as the generalized dolphin soaring problem (2.8), i.e., a simple calculus-of-variation problem of the isoperimetric type and its solution may accordingly be determined with the same (Lagrange multiplier) technique as discussed in relation to problem (2.8) in Section 2.3. Application of this technique to the present problem yields the result that the optimal velocity history $\hat{v}_p(x)$ (for the given average velocity $v_{av} > 0$ and the given $u_a(x)$, $x \varepsilon (0,L)$ is characterized by the relation (cf (2.12))

(3.2)

$$\bar{w}_p(\hat{v}_p(x)) - \hat{v}_p(x) \frac{d\bar{w}_p}{dv_p} (\hat{v}_p(x)) = z(v_{av}) - u_a(x)$$

where $z(v_{av})$ is a constant Lagrange multiplier value which in general will be different for different values of the average velocity v_{av} and where the bar over w_p signifies the use of extended-velocity-polar relationship (2.11). The actual value of the Lagrange multiplier $z(v_{av})$ may, as before, be determined from the subsidiary condition.

$$(3.3) \qquad \frac{v_{av}}{L} \int_0^L \frac{dx}{\hat{v}_p(x)} = 1$$

The value of the solution of the optimization problem (3.1) is the maximal average vertical velocity over the range in question and this average vertical velocity will play such an important role in the development to follow that it is given the special name <u>"optimal vertical range velocity"</u>. This optimal vertical range velocity w_{orv} may in principle be determined for any value of the average (horizontal-range) velocity $v_{av} > 0$ and the optimization problem (3.1) thus defines a relationship between it and the average (horizontal-range) velocity v_{av} thru the expression

$$(3.4) \qquad w_{orv}(v_{av}) :=$$

$$\max\left\{ \frac{v_{av}}{L} \int_0^L \frac{\bar{w}_p(v_p(x)) + u_a(x)}{v_p(x)} dx \;\middle|\; \frac{v_{av}}{L} \int_0^L \frac{dx}{v_p(x)} = 1 \right\}$$

This functional relationship, which may be plotted (cf Fig. 3b) in a way similar to the ordinary velocity polar, or the extended velocity polar (cf (2.11)), will be called the <u>optimal range velocity polar</u> or <u>ORV-polar</u> (for the given range and given vertical atmospheric velocity distribution).

The ORV-polar, as defined by (3.4), yields the result of the use of an optimal strategy for any given average (horizontal) velocity. Since any optimal strategy aimed at minimizing the amount of time to cross the range in question always results in some average (horizontal) velocity, it will be of interest to investigate the relation between this optimal strategy and the optimal strategy which yields the point of the ORV-polar for the same average (horizontal) velocity. It follows immediately then, that, as a consequence of the concavity of the original velocity polar (2.3), both strategies must be identical. The ORV-polar thus also <u>provides the results of all possible minimum-flight-time strategies</u>. It is this observation, which makes the ORV-polar into a useful and fundamental tool in the theory and practice of soaring flight strategies. In the remaining part of this chapter some interesting properties, as well as the construction of the ORV-polar in practice, will be discussed.

3.2 <u>Properties and Shape of the ORV-polar</u>

Intimately related to any point on the ORV-polar is the value of the Lagrange multiplier $z(v_{av})$ which determines the optimal velocity history $\hat{v}_p(x)$, $x \varepsilon (0,L)$ which produces the horizontal and vertical velocity range in question.

It turns out (and that is the key to the practical usefulness of the ORV-polar) that these z-values also play a role in the geometric characterization of the ORV-polar itself. To be precise, it can be shown that as a result of the definition (3.4) the derivative of the ORV-polar satisfies the relationship

(3.5)

$$\frac{dw_{orv}}{dv_{av}}(v_{av}) = - \frac{z(v_{av}) - w_{orv}(v_{av})}{v_{av}} \qquad (v_{av} > 0)$$

The proof of this derivative property of the ORV-polar requires some mathematical reasoning which falls outside the scope of the present discussion. The proof is for that reason deferred to Appendix A. At this point it is of more interest to remark that the derivative property implies for the ORV-polar a relationship which is similar to the McCredy relation (2.4) for regular velocity polars, to wit the relation

$$(3.6) \qquad w_{orv}(v_{av}) - v_{av}\frac{dw_{orv}}{dv_{av}}(v_{av}) = z(v_{av})$$

A sketch of the geometric implication of this relation is given in Fig. 3b.

The derivative property (3.5) illustrates the importance of the role of the Lagrange Multiplier values $z(v_{av})$ for the construction of the ORV-polar. In view of that role some inequalities which govern the relation between these z-values and the average velocity v_{av} will be given some attention before more details about the shape of the ORV-polar are discussed.

So that the ORV-polar can be defined for arbitrary (positive) average (horizontal) velocities smaller than the velocity $v_{p,mind}$ corresponding to the minimum sink rate $w_{p,max}$ of the sailplane (cf Fig. 2), one should assume the validity of extended velocity polar relationship of the form (2.11) as discussed in Section 2.3. Observing that it agrees with the usual practical situation to also assume strict concavity of the original velocity polar of the sailplane, the following relations (cf Fig. 2) will hold for the original extended velocity polar (2.11)

$$(3.7) \qquad \bar{w}_p(v_p) - v_p\frac{d\bar{w}_p}{dv_p}(v_p) = w_{p,max}$$

for $0 < v_p \le v_{p,mind}$ $> w_{p,max}$ for $v_p > v_{p,mind}$

and

$$(3.8) \qquad \bar{w}_p(v_{p,2}) - v_{p,2}\frac{d\bar{w}_p}{dv_p}(v_{p,2}) >$$

$$\bar{w}_p(v_{p,1}) - v_{p,1}\frac{d\bar{w}_p}{dv_p}(v_{p,1}) \Rightarrow v_{p,2} > v_{p,1}$$

Combination of the first relation (3.7) with the observation that the optimality condition (3.2) which determines the optimal velocity history $\hat{v}_p(x)$, $x \in (0,L)$, requires that

$$\bar{w}_p(\hat{v}_p(x)) - \hat{v}_p(x)\frac{d\bar{w}_p}{dv_p}(\hat{v}_p(x)) = z(v_{av}) - u_a(x)$$

leads for any $x \in (0,L)$ to the inequality

$$z(v_{av}) - u_a(x) \ge w_{p,max}$$

and hence, to a lower bound for the Lagrange multiplier value

$$(3.9) \qquad z(v_{av}) \ge w_{p,max} + u_{a,max} =: z_{mr}$$

where

$$(3.10) \qquad u_{a,max} := \max\{u_a(x) \mid x \in [0,L]\}$$

Combination of the second inequality relation (3.8) with the optimality condition (3.2) results in a similar implication

$$z(v_{av,2}) > z(v_{av,1}) \Rightarrow \hat{v}_{p,2}(x) > \hat{v}_{p,1}(x)$$

which relates any pair of nonidentical Lagrange multiplier values to the corresponding pair of optimal velocities. Since this last implication should hold for any $x \in (0,L)$, the following implication is an immediate consequence

$$(3.11) \qquad z(v_{av,2}) > z(v_{av,1}) \Rightarrow v_{av,2} > v_{av,1}$$

The two relations (3.9) and (3.11), the derivative property (3.5) and an important property of the optimal strategy, to be discussed in the next paragraph, together determine the general shape of the ORV-polar. This consists of a linear part (cf Fig. 3b) in the lower average-velocity range, which is mainly determined by the lower bound (3.9) of the Lagrange multiplier value, and a concave part which is determined by the relation (3.11).

With respect to the optimal velocity strategies $\hat{v}_p(x)$, $x \in (0,L)$ which produce points of the ORV-polar in the lower average velocity range, an important observation can be made which is strongly related to the assumption of an extended velocity polar relationship as expressed by (2.11). This observation, which is also of substantial importance for the practical implementation of the optimal solution, is that an optimal velocity history $\hat{v}_p(x)$, $x \in (0,L)$, can only contain at some point $x \in (0,L)$ a local (horizontal) velocity $\hat{v}_p(x)$ smaller than $v_{p,mind}$ when the corresponding Lagrange multiplier value $z(v_{av})$ is equal to its lower bound z_{mr} and when, in addition to that, at the point x the vertical atmospheric velocity $u_a(x)$ attains its maximum value $u_{a,max}$ (3.10).

The reason for this property follows from the fact that substitution of the extended polar relationship (3.7) into the optimality condition (3.2) results in the requirement that when $\hat{v}_p(x) \le v_{p,mind}$

$$z(v_{av}) - u_a(x) = w_{p,max}$$

and this equality can in view of the earlier derived lower bound for $z(v_{av})$ (3.9) only be satisfied if at the point $x \in (0,L)$

$$u_a(x) = u_{a,max}$$

An interesting practical consequence of this discussion is the rule that circling or S-ing will only be optimal when executed in points $x \in (0,L)$ where the vertical atmospheric velocity attains its maximum value[10]. For the ORV-polar this result implies that the optimal vertical range of velocities in the region of the small average velocities are the result of optimal strategies which consist of circling or S-ing in locations where the extreme vertical atmospheric velocities are present combined with a straight flight with an optimal velocity history corresponding to the lower bound z_{mr} (3.9) of the Lagrange multiplier value.

Accordingly, the ORV-polar in this region of small average velocities consists of a straight line connecting the point $(0,z_{mr})$ on the vertical axis with the minimum-straight-flight or MSF-point of the ORV-polar (cf Fig. 3b) which point, with coordinates ($v_{av,msf}$, $w_{orv,msf}$), is the result of the optimal velocity history corresponding to the Lagrange multiplier value z_{mr}. This point, which owes its name to the fact that it is the "first" point of the ORV-polar (i.e., with the lowest average velocity) realized by an optimal velocity history without circling or S-ing, is without doubt one of the most important points of the ORV-polar. As such it should preferably be one of the first points to determine in practical applications.

The preceding discussion is also of importance for the appreciation of the S-ing mode strategy put forward by Metzger and Hedrick[8] and discussed in Section 2.3. To be precise, it may be deduced that one can always replace an S-ing strategy by a strategy consisting of circling at some location x where $u_a(x) = u_{a,max}$ combined with a straight flight with horizontal velocity $v_{p,mind}$ over the other parts of the range where the relation $u_a(x) = u_{a,max}$ holds. This result implies in particular the important practical conclusion that the S-ing-mode, if optimal, is never the only optimal strategy. For theoretical purposes one can thus ignore the S-ing mode and instead restrict oneself to two flying modes, to wit a) the pure dolphin flying mode consisting of straight flight without circling and b) the regular) McCready flying mode consisting of stretches of straight flight interchanged with circling in locations with extreme vertical atmospheric velocities. It will be clear that the point of the ORV-polar which serves as border point of the regions where either of these two different flying modes is optimal, is the minimal-straight-flight or MSF-point defined above.

3.3 The Use of the ORV-polar in Theory and Practice

The ORV-polar as discussed in the preceding two sections was defined to provide all information to optimally travel over a given range with given vertical atmospheric velocity distribution with any desired average horizontal velocity. Thanks to the derivative property (3.5) of the ORV-polar, the procedure in case of a given ORV-polar and a given average velocity is a simple one: with the derivative property relation (3.6) the Lagrange multiplier value $z(v_{av})$ can be evaluated immediately (in practice possibly even by graphical means) and this Lagrange multiplier value determines via the optimality condition (3.2) the (optimal) velocity history of the straight flight portion of the optimal trajectory.

In practice, the Lagrange multiplier value $z(v_{av})$ found may, as a result of the similarity between relation (3.2) and relation (2.7), be used directly as a McCready-ring setting for use in connection with a McCready-ring or "Sollfahrtgeber" (cf Section 2.2). The pilot may thus generate the optimal velocity history in the usual way. In connection with this observation, the words Lagrange multiplier values and McCready-ring-settings will be used interchangeably for the $z(v_{av})$ values in the rest of this paper.

In order to make use of the ORV-polar it is not necessary to specify ahead of time the numerical value of the average velocity to be considered. To the contrary, the ORV-polar itself provides a very useful means for determining for any given optimization objective the correspoiding optimal average velocity v_{av} to travel over the range under consideration. In particular, the availability of an ORV-polar for a given range with given vertical atmospheric velocity distribution makes it possible to determine the optimal average velocity v_{av} (and therewith, as discussed, the Lagrange multiplier value $z(v_{av})$ and the optimal velocity history $\hat{v}_p(x)$, $x \in (0,L)$ to travel in an optimal way over the range considered at any overall glide or climb angle that is feasible under the prevailing conditions.

Of most interest in practical situations is, of course, the optimal average velocity (with corresponding Lagrange multiplier value and optimal velocity history) for crossing the range with no altitude loss or gain. The average velocity which yields this result is given by the intersection point of the ORV-polar with the horizontal axis, which point for that reason is called the zero (-altitude) -loss or ZL-point. The corresponding average velocity is denoted as $v_{av,zl}$ and is determined by the condition that (cf Fig. 3b)

$$(3.12)\qquad w_{orv}(v_{av,zl}) = 0$$

The optimal velocity history which corresponds to the ZL-point on the ORV-polar is just the solution of the generalized dolphin soaring problem (2.8) for $\gamma = 0$. The corresponding Lagrange multiplier value with which the optimal velocity history may be generated (cf (3.2)) and which itself is given by (cf (3.6))

(3.13)

$$z_{opt} := z(v_{av,zl}) := - v_{av,zl} \frac{dw_{orv}}{dv_{av}}(v_{av,zl})$$

is called the optimal McCready-ring-setting for the range in question. For most practical purposes the knowledge of this optimal McCready ring setting is as good as the knowledge of the complete ORV-polar.

Depending on whether the ZL-point is situated on the straight line (or McCready) segment of the ORV-polar or the curved (or dolphin-flying) part of the ORV-polar the corresponding optimal trajectory represents either the McCready-flying mode for which

$$(3.14)\qquad z_{opt} = z_{mr}$$

or the pure dolphin-flying mode for which

$$(3.15)\qquad z_{opt} > z_{mr}$$

A fast way to determine which of these two modes apply is to evaluate the optimal-vertical range-velocity $w_{orv,msf}$ corresponding to the MSF-point (cf Fig. 3) of the ORV-polar. Whenever this optimal vertical range velocity $w_{orv,msf}$ is nonpositive, the ZL-point lies on the straight line segment of the ORV-polar and the McCready-flying mode applies, i.e.,

$$(3.16)\qquad w_{orv,msf} \leq 0 \rightarrow z_{opt} = z_{mr} \text{ (McCready mode)}$$

Otherwise the ZL-point lies on the curved segment of the ORV-polar and the pure-dolphin flying mode applies

$$(3.17)\qquad w_{orv,msf} > 0 \rightarrow z_{opt} > z_{mr} \text{ (pure dolphin mode)}$$

Conditions (3.16) and (3.17) thus illustrate the importance of the knowledge of the location of the MSF-point in actual practice.

4. THE CONSTRUCTION OF THE ORV-POLAR

4.1 General Procedure

The ORV-polar for a given range and vertical atmospheric velocity distribution could in principle be determined by solving for each average (horizontal) velocity v_{av} the optimization problem (3.4) by which the ORV-polar was defined in Section 3.1. In practice, however, simpler precedures may be used which are based on the special properties of the ORV-polar discussed in Section 3.2.

In particular, use may be made of the property that for average velocities smaller than the average velocity corresponding to the minimum-straight-flight or MSF-point, the ORV-polar consists of a straight line connecting the point $(0, z_{mr})$ on the vertical axis and the MSF-point with coordinates $(v_{av,msf}, w_{orv,msf})$. For average velocities equal to or greater than the average velocity corresponding to the MSF-point, the points of the ORV-polar may be determined by evaluating the integrals

$$(4.1)\qquad T(z) = \int_0^L \frac{dx}{v_p(x)}$$

and

$$(4.2)\qquad \Delta h(z) = \int_0^L \frac{\bar{w}_p(v_p(x)) + u_a(x)}{v_p(x)}\, dx$$

in which expressions optimal velocity histories $v_p(x)$ are to be substituted, which are generated for fixed values of the Lagrange multiplier $z \geq z_{mr}$ using the optimality condition (3.2) with z replacing $z(v_{av})$

$$(4.3)\qquad \bar{w}_p(v_p(x)) - v_p(x) \frac{d\bar{w}_p}{dv_p}(v_p(x)) = z - u_a(x)$$

The corresponding points of the ORV-polar then follow directly from

$$(4.4)\qquad v_{av}(z) = L/T(z)$$

and

$$(4.5)\qquad w_{orv}(z) = \Delta h(z)/T(z)$$

The actual calculation of these quantities can be easily performed on a digital computer as soon as some polynomial approximation of the velocity polar

$$(4.6)\qquad w_p(v_p) \cong \sum_{k=k_{min}}^{k_{max}} c_k v_p^k$$

and a polynomial approximation of the McCready curve[10,11].

$$(4.7)\qquad z_p(v_p) = w_p(v_p) - v_p \frac{dw_p}{dv_p}(v_p) \cong \sum_{k=k_{min}}^{k_{max}} (1-k)\, c_k v_p^k$$

is available. Necessary for the evaluation of the optimal velocity history from the optimality condition (3.2) is the inverse of this last function

$$(4.8)\qquad v_p = z_p^{\leftarrow}(z) =: v_p(z)$$

Several methods may be used to determine an approximation for this inverse function, which will be assumed to be given in the discussions to follow.

The prefered procedure for the determination of the ORV-polar thus consists of selecting successively increasing values of $z \geq z_{mr}$ starting off with $z = z_{mr}$ and to determine for each of them the corresponding point of the ORV-polar. In case a point of the ORV-polar corresponding to a particular average velocity v_{av} is desired then some iterative procedure to determine the Lagrange multiplier $z(v_{av})$ which produces the given average velocity will in general be required. Only in the case where the average velocity in question is smaller than the average velocity corresponding to the minimum-straight-flight point can use be made of the local linearity of the ORV-polar to circumvent an iterative procedure.

4.2. Adaptation of an ORV-polar in Case of Thermals

The usefulness of the ORV-polar for practical optimization purposes is very much enhanced by the ease with which existing ORV-polars may be adapted in case thermals are present at an initial and/or end point and, even more general, the ease with which ORV-polars over subsequent ranges may be combined with ORV-polars over larger ranges and, in the ideal case, even to the ORV-polar pertaining to the total range covered by the sailplane on its cross-country flight. The two typical situations: 1) the adaptation of an existing polar to account for a thermal at an initial and/or end point and 2) the synthesis of two similar ORV-polars over subsequent ranges, will be discussed in some detail in this and the next section.

The determination of the change of the ORV-polar when a thermal at one end is to be taken into account is of much conceptual interest. As a result of the thermal at the end point the maximal value of the sum of the vertical velocity of the atmosphere and the vertical velocity of the sailplane over the "enlarged" range will in general no longer be equal to (3.9)

$$z_{mr} := w_{p,max} + u_{a,max}$$

but instead will become equal to the net rate-of-climb in the thermal (2.1)

$$z_{th} := w_{p,max} + u_{th}$$

where u_{th} is the vertical atmospheric velocity in the thermal. Following the rules discussed in the preceding sections, the new ORV-polar will contain a new straight line connecting the point $(0, z_{th})$ on the vertical axis with a new MSF-point (cf preceding sections) on the original ORV-polar, which point is characterized by the fact that the corresponding Lagrange multiplier value satisfies

$$z(v_{av,msf}) = \max\{z_{mr}, z_{th}\} = z_{th}$$

The optimal strategy for each point on this straight line segment consists of circling within the thermal followed (or preceded) by a straight flight with an optimal velocity history corresponding to the Lagrange multiplier value or McCready-ring setting equal to z_{th} (3.12). For average velocities larger than the average velocity $v_{av,msf}$ of the MSF point the original ORV-polar is not altered if the horizontal dimensions of the thermal may be assumed to be very small relative to length of the range.

It is clear that the condition of the new ORV-polar in this last case agrees with the traditional graphical construction of the solution of classical McCready problem (cf Section 2.1). This similarity is not accidental: The ORV-polar of a sailplane flying over a range with completely still air is precisely the original (extended) velocity polar of the sailplane and seen in that light both constructions are identical. The main point to be observed here is that the same construction as before may also be applied to more general ORV-polars which for the sake of this and similar constructions in the next section may be treated as if they were no more than regular (extended) velocity polars. This aspect in particular is a very strong point in favor of the use of the ORV-polar-concept in the theory and the practice of the optimization of sailplane trajectories.

4.3 The Synthesis of Two or More ORV-polars

Of much interest for theoretical as well as practical purposes is the procedure for the synthesis of two or more ORV-polars to yield one resulting ORV-polar which corresponds to the combination of the ranges. The keys to this procedure are two observations which directly relate to the porperties of the ORV-polar discussed in Section 3.2. The first observation is that the lower bound on the McCready-ring settings corresponding to the new ORV-polar will be the largest value of the net rate-of-climb over the range (cf (3.9)), i.e., the sum of the maximum value of the vertical atmospheric velocity over the range and the maximum value of the vertical velocity of the sailplane. Evidently, this maximum value will be equal to the maximum of the minimal McCready-ring settings $z_{mr,i}$ of the contributing ORV-polars, i.e.,

$$z_{mr,s} := \max_i \, [z_{mr,i}] \tag{4.9}$$

The second observation is that at any point of the resulting ORV-polar there will correspond an optimal velocity history $v_p(x),\ x \in [0, \Sigma_i L_i]$ which may be determined by the substitution of one Lagrange multiplier value $z \geq z_{mrs}$ in the optimality condition (3.2). Again it will be immediately clear that such an optimal velocity history will be nothing else than the sequence of the optimal velocity histories over the subsequent ranges corresponding to the same Lagrange multiplier value.

A direct consequence of these two observations is that for all values z of the Lagrange multiplier or McCready ring-setting larger than or equal to the minimal McCready ring-setting $z_{mr,s}$ the corresponding point of the resulting ORV-polar can be found by combining the average horizontal velocities $v_{av,i}(z)$ and optimal vertical range velocities $w_{orv,i}(z)$ corresponding to the particular value of z following the straight forward expressions

$$v_{av,s}(z) := \sum_{i=1}^{m} \left(\frac{L_i}{v_{av,i}(z)} \Big/ \sum_{j=1}^{m} \frac{L_j}{v_{av,j}(z)}\right) v_{av,i}(z) \tag{4.10}$$

and

$$w_{orv,s}(z) := \sum_{i=1}^{m} \left(\frac{L_i}{v_{av,i}(z)} \Big/ \sum_{j=1}^{m} \frac{L_j}{v_{av,j}(z)}\right) w_{orv,i}(z) \tag{4.11}$$

which expressions for the case that m = 2 reduce to

$$v_{av,s}(z) := \frac{L_1 v_{av_2}(z) v_{av_1}(z)}{L_1 v_{av_2}(z) + L_2 v_{av_1}(z)} + \frac{L_2 v_{av_1}(z)\, v_{av_2}(z)}{L_1 v_{av_2}(z) + L_2 v_{av_1}(z)} \tag{4.12}$$

$$= \frac{(L_1+L_2)\, v_{av,1}(z) v_{av,2}(z)}{L_1 v_{av,2}(z) + L_2 v_{av,1}(z)}$$

$$w_{orv,s}(z) := \frac{L_1 v_{av,2}(z)}{L_1 v_{av,2}(z) + L_2 v_{av,1}(z)} w_{orv,1}(z) + \frac{L_2 v_{av,1}(z)}{L_1 v_{av,2}(z) + L_2 v_{av,1}(z)} w_{orv,2}(z) \tag{4.13}$$

Values of z smaller than the overall minimal McCready-ring-setting $z_{mr,s}$ (4.9) which generated optimal velocity histories over the original parts of the resulting range no longer do so. For average velocities smaller than the average velocity $v_{av}(z_{mr,s})$ for minimum straight flight over the resulting range the corresponding points of the resulting ORV-polar lie on the line connecting the point $(0, z_{mr,s})$ on the vertical axis and the new MSF-point $(v_{av}(z_{mr,s}), w_{orv}(z_{mr,s}))$. For the practice of optimal dolphin flight this last result implies the nowadays well known rule[10] that circling in order to gain missing height should in theory only be done at the location where the vertical atmospheric velocity reaches its maximum.

The expressions (4.0)-(4.13) for the resulting average velocities and the resulting optimal vertical range velocities corresponding to a particular $z \geq z_{mr,s}$ imply that the resulting range velocity vector with coordinates $(v_{av,s}(z), w_{orv,s}(z))$ is a convex combination of the original range velocity vectors. In the case that m = 2 this implies in particular that the resulting range vector lies on the line which connects the points on the original ORV-polars corresponding to the same z. A sketch of this geometric interpretation of the synthesis of two ORV-polars is given in Fig. 4. Of much interest for a possible practical use of this interpretation is the observation that the lines which connect the points (0,z) on the vertical axis with the corresponding points on the original and resulting ORV-polars cut off pieces from any vertical lines which differ from each other by a ratio equal to the ratio of lengths of the original ranges (cf Fig. 4). This geometric property, the proof of which will be given in Appendix B, paves the way for simple graphical methods for the construction of ORV-polars which result from the combination of two ORV-polars corresponding to two subsequent ranges.

5. A Practical Application: The Determination of the Optimal McCready-ring Setting in Case of a Square-wave Vertical Atmospheric Velocity Distribution

5.1 The ORV-polar for a Square Wave Thermal Model

Although it is in principle possible to determine the ORV-polar for any range and any vertical atmospheric velocity distribution, the actual calculation will in general be restricted to some simple models. In actual practice a sailplane pilot will never know the exact vertical atmospheric velocity at some locations before he arrives there. Therefore, it is much more useful for practical purposes to merely provide the pilot some guidelines based on simple models and to leave it to him to interpret the actual situation in the light of his knowledge about the optimal solutions for those simple models.

A particular model which is of interest is the general square wave vertical atmospheric velocity model, which, as advocated by Reichmann[10], does not necessarily satisfy the mass balance relation (i.e., the mass of air going up along the range does not necessarily equal the mass of air going down). In actual practice such a square wave model will approximately apply in case there are cloud streets roughly along the course of the flight. It is the maneuvering of the pilot in such circumstances which tips the air mass balance into his favor.

For the square wave model to be considered it will be assumed (see Fig. 5) that the range consists of two parts of lengths L_1 and L_2 on each of which there is a constant vertical atmospheric velocity present with strengths u_1 and u_2 for which the additional arbitrary assumption is made that $u_2 \geq u_1$. (Of course, under the prevalent assumptions, the optimal solution would not change if L_1 and L_2 would consist of a number of distinct pieces adding up in length to L_1 and L_2.) The ratio $L_2/(L_1+L_2)$ which may be considered the fraction of the range over which the hypothetical cloud street extends will be called the extension factor and will be denoted by the letter e.

The determination of the ORV-polar corresponding to this square wave vertical velocity distribution model is relatively simple once it is observed that the ORV-polar may be thought of as the result of the syntheses of the two directly available ORV-polars for the parts L_2 and L_2 of the total range (cf Fig. 6). Each of these consists of a translation in vertical direction of the original extended velocity polar, i.e., in formula form one has, respectively

$$(5.1) \quad w_{orv,1}(v_{av}) := w_{p,max} + u_1 =: z_1 \quad \text{if } v_{av} \leq v_{p,mind}$$
$$:= w_p(v_{av}) + u_1 \quad \text{if } v_{av} > v_{p,mind}$$

and

$$(5.2) \quad w_{orv,2}(v_{av}) := w_{p,max} + u_2 := z_2 \quad \text{if } v_{av} \leq v_{p,mind}$$
$$:= w_p(v_{av}) + u_2 \quad \text{if } v_{av} > v_{p,mind}$$

Particular points $(v_{av,1}(z), v_{orv,1}(z)$ and $(v_{av,2}(z), w_{orv,z}(z))$ of the original ORV-polars corresponding to values of z which satisfy respectively $z \geq z_1$ and $z \geq z_2$ may be determined directly from the inverse function $v_p(z)$ of the McCready function (cf 4.8) of the original extended velocity polar following the straight forward relations

$$(5.3) \quad \begin{aligned} v_{av,1}(z) &:= v_p(z - u_1) \\ w_{orv,1}(z) &:= w_p(v_p(z - u_1)) + u_1 \end{aligned}$$

and

$$(5.4) \quad \begin{aligned} v_{av,2}(z) &:= v_p(z - u_2) \\ w_{orv,2}(z) &:= w_p(v_p(z - u_2)) + u_2 \end{aligned}$$

In this context it should be noted that as a result of the vertical atmospheric velocities being constant, the equivalent expressions (cf 4.8): $v_{av,1}(z_1)$, $v_{av,2}(z_2)$ and $v_p(w_{p,max})$ do not represent unique velocities but instead the whole range of velocities between 0 and $v_{p,mind}$.

The coordinates of the MSF-point of the new ORV-polar may readily be determined once it is observed that for the square wave model under consideration

$$(5.5) \quad z_{mr} = \max [z_1, z_2] = z_2$$

and that for the "minimum-straight-flight" trajectory

$$(5.6) \quad v_{av,1}(z_2) = v_p(z_2 - u_1) \quad , \quad v_{av,2}(z_2) = v_{p,mind}$$

Substitution of the expressions (5.3) and (5.4) evaluated for $z = z_2$ into the previously derived expressions (4.12) and (4.13) for the coordinates of an ORV-polar produced by synthesis of two ORV-polars immediately results in the desired quantities

$$(5.7) \quad v_{av,msf} = \frac{(L_1 + L_2)v_{p,mind}\, v_p(z_2 - u_1)}{L_1\, v_{p,mind} + L_2\, v_p(z_2 - u_1)}$$

and

$$(5.8) \quad w_{orv,msf} = \frac{L_1\, v_{p,mind}\, [w_p(v_p(z_2 - u_1)) + u_1] + L_2\, v_p(z_2 - u_1) z_2}{L_1\, v_{p,mind} + L_2\, v_p(z_2 - u_1)}$$

Coordinates of points of the new ORV-polar which correspond to Lagrange multiplier values z larger than $z_{mr} = z_2$ follow in a similar way from the expressions

$$(5.9) \qquad v_{av}(z) = \frac{(L_1 + L_2) v_p(z - u_2)\, v_p(z - u_1)}{L_1\, v_p(z - u_2) + L_2\, v_p(z - u_1)}$$

and

$$(5.10) \qquad w_{orv}(z) = \frac{L_1 v_p(z-u_2)[w_p(v_p(z-u_1))+u_1] + L_2 v_p(z-u_1)[w_p(v_p(z-u_2))+u_2]}{L_1 v_p(z-u_2) + L_2 v_p(z-u_1)}$$

It should be noted that with the geometric property (cf Fig. 4) of the lines connecting the common point (0,z) on the vertical axis with the corresponding points on the original and resulting ORV-polars, as discussed in Section 4.3 and proved in Appendix B, it is quite simple to construct the resulting ORV-polar by purely graphical means for any given values of u_1, u_2 and $e = L_2/(L_1+L_2)$. A particular application of this will be discussed in the next section.

5.2. The Optimal McCready-Ring-Setting for a Square Wave Thermal Model

As discussed in Section 3.3 the quantity of most interest in connection with the ORV-polar for practical purposes is the optimal McCready-ring setting z_{opt} (3.13) which was defined as the McCready-ring setting which generates the ZL-point of the ORV-polar, i.e., the intersection point of the ORV-polar with the horizontal axis. The way to determine this optimal McCready-ring setting depends on whether the ZL-point lies on the straight line segment of the ORV-polar or on the curved segment thereof and this in turn depends on whether the MSF-point (cf Section 3.3) lies above or below the horizontal axis or, equivalently, whether the optimal vertical range velocity $w_{orv,msf}$ corresponding to the MSF-point is positive or negative. In the case of a square-wave thermal model this latter velocity is given by (5.8) so that the determination of the optimal McCready-ring setting depends on the inequality

$$(5.11) \qquad L_1 v_{p,mind}[w_p(v_p(z_2-u_1))+u_1] + L_2 v_p(z_2-u_1) z_2 \leq 0$$

As long as this inequality is satisfied the optimal McCready-ring setting is equal to

$$(5.12) \qquad z_{opt} = z_2 = u_2 + w_{p,max}$$

and the corresponding average (horizontal) velocity equal to (cf Fig. 3)

$$(5.13) \qquad v_{av,zl} = \left(\frac{z_2}{z_2 - w_{orv,msf}}\right) v_{av,msf}$$

which expression with (5.7) and (5.8) results in

$$(5.14) \qquad v_{av,zl} = \frac{L_1+L_2}{L_1} \cdot \frac{z_2}{z_2 - [w_p(v_p(z_2-u_1))+u_1]} \cdot v_p(z_2-u_1)$$

The corresponding strategy in this case will be of the <u>McCready type</u>, i. e., one should travel over part L_1 (cf Fig. 5) of the range with the velocity $v_p(z_2-u_1)$ corresponding to an expected net rate of climb z_2 (under the cloud street), and over part L_2 of the range with horizontal velocity $v_{p,mind}$.

Finally, one should regain the missing altitude by circling at the end of the range with net rate of climb z_2. The strategy of flying straight over part L_2 of the range at the horizontal velocity $v_{p,mind}$ followed by circling at the end of the range is of course equivalent to "S-ing" over part L_2 of the range with an average velocity $v_{av}(z_2)$ which follows from setting $w_{orv}(z_2)$ as given by (4.13) equal to zero, i.e.,

$$(5.15) \qquad v_{av,2}(z_2) = -\frac{L_2}{L_1} \cdot \frac{z_2}{w_p(v_p(z_2-u_1))+u_1} \cdot v_p(z_2-u_1)$$

A graphical illustration of this and the preceding expression (5.14) is presented in Fig.6.

In case inequality (5.11) is not satisfied, then the optimal McCready-ring setting has to be determined as the value $z > z_2$ for which the optimal vertical range velocity (4.13) or (5.10) is equal to zero, i.e., the solution of the equation

$$(5.16) \qquad L_1 v_p(z-u_2)[w_p(v_p(z-u_1))+u_1] + L_2 v_p(z-u_1)[w_p(v_p(z-u_2))+u_2] = 0$$

For the solution of this equation some iterative procedure will be nexessary in general. The optimal strategy in this case will be of the <u>pure-dolphin-type</u>.

5.3 Plots of Z_{opt} vs e for Square Wave Thermal Models

Given the means for solving for the optimal McCready-ring setting in case of a square wave vertical atmospheric velocity model with given values of u_1, u_2 and $e = L_2/L_1+L_2$, a very useful piece of information for actual flying practice may be generated in the form of a plot of the optimal McCready ring settings z_{opt} as a function of the extension factor e for various combinations of u_1 and u_2. Plots of this kind have already been produced by Metzger and Hedrick[8], however, given the theory of the ORV-polars as discussed, the information necessary for these plots may be generated without an iterative procedure and, if desired, even by simple graphical means. Because of the practical importance of this implementation of the theory, the generation of plots of z_{opt} vs e will be discussed in some more detail below. The numberical results for a particular sailplane will be given in the next section.

For given fixed values of u_1 and u_2 the plot of the optimal McCready-ring setting z_{opt} as a function of the (cloud street) extension factor e will in general consist of two parts: a constant McCready-ring setting corresponding to a McCready-type optimal strategy for small values of e and a roughly linearly increasing part, corresponding to pure dolphin-type optimal strategy for higher values of e. The value of the former constant optimal McCready-ring setting is given by

$$(5.12) \qquad z_{opt} = z_2 = w_{p,max} + u_2$$

Points of the increasing part of the plot may be generated in a straight forward way by solving equation (5.9) for given values of u_1, u_2 for e given z instead of for z given e. This is feasible by straight forward evaluation of (5.16) written in terms of e

$$(1-e)v_p(z-u_2)[w_p(v_p(z-u_1))+u_1] +$$
$$e\, v_p(z-u_1)[w_p(v_p(z-u_2)+u_2] = 0$$

which leads to the explicit expression

$$(5.17) \qquad e(u_1,u_2,z) =$$
$$\frac{v_p(z-u_2)[w_p(v_p(z-u_1))+u_1]}{v_p(z-u_1)[w_p(v_p(z-u_2))+u_2] - v_p(z-u_2)[w_p(v_p(z-u_1)+u_1]}$$

Evaluation of this expression for $z_{opt} = z_2$ with the appropriate substitutions (5.2) and (5.5) provides immediately the <u>corner point</u> or <u>break point</u> $e_c(u_1,u_2)$ of the z_{opt} vs e plot

$$(5.18) \qquad e_c(u_1,u_2) = e(u_1,u_2,z_2) =$$
$$-\frac{v_{p,mind}[w_p(v_p(z_2-u_1)+u_1]}{v_p(z_2-u_1)z_2-v_{p,mind}[w_p(v_p(z_2-u_1))+u_1]}$$

It may be noted that the value given by this formula provides the minimal cloud street extension factor for the realization of a pure dolphin-flight strategy (for the given values of u_1 and u_2).

The maximal value of the optimal McCready-ring setting z_{opt} for given values of u_1 and u_2 will of course result when e = 1, i.e., when the cloud street extends over the whole range. The value of it may formally be found by the solution of the equation that results when $L_1 = 0$ is substituted into (5.6), i.e.,

$$(5.19) \qquad w_p(v_p(z-u_2)) + u_2 = w_{orv,2}(z) = 0$$

The solution of this equation in principle requires, just as in the other cases of fixed values of e, an iterative procedure. In this special case, however, this iterative procedure amounts to no more than the generation of the inversion of velocity polar relationship

$$(5.20) \qquad w_p(v_p(z-u_2)) = -u_2$$

Consequently, this particular maximal optimal McCready-ring setting may in practice be determined by the same graphical procedure as discussed in Section 2.1.

It may be noted that the initial point, as well as the final point of the plot of z_{opt} vs e are exclusively determined by the value of the largest of the two constant vertical atmospheric velocities over the range, i.e., the value of u_2 over the second part L_2 (cf Fig. 5) of the range. The value of u_1 of the vertical atmospheric velocity over the first part of the range does, in combination with the value u_2 determine the location of the corner point $e_c(u_1,u_2)$ of the plot of z_{opt} ve e and therewith the average slope of the plot in the pure dolphin flight region past the corner point. An example of this behaviour is given in Figures 7-8 which present the numerical results which were calculated for a particular sailplane type.

As a final observation it may be noted that the plot of z_{opt} ve e in case of a square wave vertical atmospheric velocity distribution model was evaluated with as only information the velocity polar $w_p(v_p)$ and the inverse $v_p(z)$ (4.8) of the regular McCready function. No integration procedure or other complicated procedures were required and this in particular implies that the same effort by any sailplane pilot who has the two mentioned pieces of information available. With the use of the geometric property discussed in Section 4.3 and proved in Appendix B the data for the z_{opt} ve e plot may even be generated by a pure graphical method in much the same way as in the years past the data for McCready rings have been generated by a great number of pilots.

5.4 Numerical Results for an LS-3 Sailplane

In order to provide a numerical example of the results presented in this chapter, calculations of the optimal McCready-ring settings z_{opt} as a function of the cloud street extension factor e in case of a square wave thermal model were carried out for an LS-3 sailplane which is a representative specimen of the modern "unrestricted 15-m class" or "racing class" of sailplanes. The numerical data for this particular sailplane was taken from calculated velocity polars (corresponding to wing loadings or respectively 33 kg/m^2 and 45 kg/m^2) furnished by the manufacturer. From each of these velocity polars 20 readings were taken and fed into a computer program for least squares polynomial approximation. For the velocity polar corresponding to a wing loading of 33 kg/m^2 the following polynomial turned out to be a reasonably accurate approximation (with a maximum relative error of 1.58%)

$$\begin{aligned} w_p(v_p) = &- 0.144534\ (v_p/40)^{-2} + 2.138253\ (v_p/40)^{-1} \\ &- 7.847412 \qquad\qquad + 14.014615\ (v_p/40) \\ &- 11.318253\ (v_p/40)^2 + 4.389605\ (v_p/40)^3 \end{aligned}$$

where v_p and w_p are both expressed in m/sec.

Similarly, the velocity polar corresponding to a wing loading of 45 kg/m^2 turned out to be reasonably accuratey (with a maximum relative error of 1.57%) approximated by the polynomial

$$\begin{aligned} w_p(v_p) = &- 1.45118\ (v_p/40)^{-2} + 9.510116\ (v_p/40)^{-1} \\ &-22.681673 \qquad\qquad + 27.045180\ (v_p/40) \\ &-15.558411\ (v_p/40)^2 + 4.223420\ (v_p/40)^3 \end{aligned}$$

where again w_p and v_p are both expressed in m/sec.

Using these polynomial approximations of the velocity polars two different sets of plots of z_{opt} ve e were evaluated for each of the two wing loadings considered. Figures 7 and 8 provide the detailed results for a number of different values of vertical atmospheric velocity u_2 over the second part L_2 of the range and no vertical atmospheric velocity (i.e., $u_1 = 0$) over the first part L_1 over the range. Also shown in the same plots are the results for different values of the vertical atmospheric velocity u_1 over the first part of the range. The plots are shown in Figures 7 and 8.

6. CONCLUDING REMARKS

The idea of studying the set of solutions of the optimization problem maximizing the average vertical velocity for given average horizontal velocities over the given range with given vertical atmospheric velocity distribution has been proven profitable for both the theory and the practice of sailplane flight trajectory optimization problems. The resulting theory, with the ORV-polar as fundamental item, not only unified the current theory on the solutions of the McCready problem and the generalized dolphin soaring problem, but also provided a simple means to determine the optimal strategy for the generalized dolphin soaring problem in actual practice.

Of most importance for the theory was the discovery that the tangent to the ORV-polar in some points cuts off a piece of the vertical axis which is just equal to the Lagrange multiplier value or McCready-ring setting by which the optimal velocity history corresponding to the particular point can be generated. Not only did this property determine the general shape of the ORV-polar, it also formed the basis for the easy construction of the ORV-polar. As shown, the property proved particularly fruitful for the construction of the ORV-polar by means of an adaptation of a given ORV-polar for a thermal at some point of the range or by means of a syntheses of two known ORV-polars. In this context also, a direct link was laid between the well-known McCready-theory and the ORV-polar theory presented here.

For the theory of sailplane flight trajectory optimization the simplicity of the ORV-polar-concept was shown to be very useful. With it a number of rules for optimal dolphin soaring earlier mentioned in the literature could be explained very readily. This related in particular to the rules that: 1) the optimal strategy for dolphin soaring is, as in the case of the McCready-problem, determined by one and no more than one McCready-ring setting; 2) possible missing height should be regained by circling only at those points of the range where the vertical atmospheric velocity has its maximum over the range; and that 3) one should use a higher McCready-ring setting whenever an overall height gain might result in a cloud street situation. A nice aspect of the ORV-polar theory was furthermore that it provided a direct and easily computable answer to the question of what McCready-ring setting to use, namely the value at which the tangent to the ORV-polar in the intersection point of the ORV-polar with the horizontal azis cuts off from the vertical axis. Still another rule for optimal dolphin soaring which also followed immediately from the ORV-polar was the rule that an "S-ing" strategy, when optimal, is never the unique optimal strategy. This rule yields the conclusion that from a theoretical point of view, "S-ing" strategies do not have to be considered.

For the practice of the optimization of sailplane trajectories, the ORV-polar concept was should to provide a means by which for simple thermal models, such as Reichmann's square wave model, the optimal McCready-ring settings may be determined by a simple graphical procedure resembling the procedure by which the McCready-ring data are usually obtained. In particular, plots of optimal McCready-ring settings (cf Figures 7-8) versus the cloud street extension factors may be constructed with the aid of no more than a graph of the velocity polar, a ruler and a pencil, i.e., by any pilot who is capable of determining the data for a McCready-ring.

Of course, the ORV-polar theory is by no means the final answer to the sailplane flight trajectory optimization problem. The great number of simplifying assumptions, such as for example the complete knowledge of the vertical velocity distribution ahead of time, the independence of the vertical atmospheric velocity distributions and the aerodynamic equilibrium of the height and the assumption that circling will not result in a smaller vertical velocity, are all sources for discrepancies between the optimal strategies in theory and practice. However, as usual, the theoretical results do provide more insight as to what should be done in a practical situation. The extra insight may prove a further detailed study of the ORV-polar concept without one or more of the simplifying assumptions worth while in the future.

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APPENDIX A:

PROOF OF THE DERIVATIVE PROPERTY (3.5)

In Section 3.1 the optimal vertical range velocity w_{orv} was defined as a function of the average (horizontal range) velocity v_{av} as the value of the solution of the optimization problem (cf (3.4))

$$\text{(A.1)}\quad w_{orv}(v_{av}) = \max\left\{\frac{v_{av}}{L}\int_0^L \frac{w_p(v_p(x))+u_a(x)}{v_p(x)}\,dx \;\middle|\; \frac{v_{av}}{L}\int_0^L \frac{dx}{v_p(x)} = 1\right\}$$

In this appendix it will be shown that the derivative of this function $w_{orv}(v_{av})$ with respect to its argument v_{av} is given by (3.5)

$$\text{(A.2)}\quad \frac{dw_{orv}}{dv_{av}}(v_{av}) = -\frac{z(v_{av}) - w_{orv}(v_{av})}{v_{av}} \qquad (v_{av} > 0)$$

where $z(v_{av})$ is the value of the constant Lagrange multiplier for the constrained optimization problem (A.1).

Starting point of the point is the observation that the definition (A.1) for $v_{av} > 0$ is equivalent to

$$w_{orv}(v_{av}) = \frac{v_{av}}{L}\max\left\{\int_0^L \frac{w_p(v_p(x))+u_a(x)}{v_p(x)}\,dx \;\middle|\; \int_0^L \frac{dx}{v_p(x)} = \frac{L}{v_{av}}\right\}$$

and hence also to

$$\text{(A.3)}\quad w_{orv}(v_{av}) = \frac{v_{av}}{L}\max\left\{\int_0^L \frac{w_p(v_p(x))+u_a(x)-z}{v_p(x)}\,dx + \frac{zL}{v_{av}} \;\middle|\; \int_0^L \frac{dx}{v_p(x)} = \frac{L}{v_{av}}\right\}$$

In connection with this last expression one may define the functional

$$\text{(A.3)} \qquad \Phi[v_p(x),z,v_{av}] := \int_0^L \frac{w_p(v_p(x)) + u_a(x) - z}{v_p(x)}dx + \frac{zL}{v_{av}}$$

and observe that the optimal vertical range velocity in terms of this functional is given by

$$\text{(A.4)} \qquad w_{orv}(v_{av}) = \frac{v_{av}}{L}\,\Phi\,[v_p(x;v_{av}),z(v_{av}),v_{av}]$$

where $v_p(x;v_{av})$ and $z(v_{av})$ are the optimal velocity history and the Lagrange multiplier respectively corresponding to solution of the optimization problem (A.1). The expression (A.4) found is no longer an optimization problem but instead an expression involving a functional which is dependent on a parameter v_{av}. Assuming smoothness, and differentiability properties as usual in the calculus-of-variations[5] the expression may be differentiated following the rules of the calculus of variations with as result the expression

$$\frac{dw_{orv}}{dv_{av}}(v_{av}) = \frac{1}{L}\,\Phi[v_p(x;v_{av}),z(v_{av}),v_{av}]$$

$$\text{(A.5)} \qquad + \frac{v_{av}}{L}\,[\int_0^L \frac{\partial F}{\partial v_p}(v_p(x;v_{av}),z(v_{av}))\,\frac{\partial v_p}{\partial v_{av}}(x;v_{av})dx$$

$$- (\int_0^L \frac{dx}{v_p(x;v_{av})} - \frac{L}{v_{av}})\,\frac{dz}{dv_{av}} - \frac{zL}{v_{av}^2}]$$

in which

$$\frac{\partial F}{\partial v_p}(v_p(x;v_{av}),z(v_{av})) :=$$

$$\frac{v_p(x;v_{av})\,\frac{dw_p}{dv_p}(v_p(x;v_{av})) - w_p(v_p(x;v_{av})) - u_a(x) + z(v_{av})}{v_p^2(x;v_{av})}$$

In view of the optimality of the solution $v_p(x;v_{av})$ it follows that as well (cf (3.2))

$$\frac{\partial F}{\partial v_p}(v_p(x;v_{av}),z(v_{av})) = 0 \qquad \text{for all } x \in [0,L]$$

as

$$\int_0^L \frac{dx}{v_p(x;v_{av})} - \frac{L}{v_{av}} = 0$$

So that the expression (A.5) reduces to

$$\frac{dw_{orv}}{dv_{av}}(v_{av}) = \frac{1}{L}\,\Phi[v_p(x;v_{av})z(v_{av}),v_{av}] - \frac{z(v_{av})}{v_{av}}$$

or with (A.4)

$$\frac{dw_{orv}}{dv_{av}}(v_{av}) = \frac{w_{orv}(v_{av}) - z(v_{av})}{v_{av}}$$

which is precisely the relation to prove.

APPENDIX B:

PROOF OF A GEOMETRIC PROPERTY OF THE ORV-POLAR CONSTRUCTION FOR A SQUARE WAVE THERMAL MODEL

In Section 4.3 expressions (cf (4.12 & 4.13)) were derived for the average (horizontal) velocity $v_{av,s}$ and the optimal vertical range velocity $w_{orv,s}$ (corresponding to some value of the McCready-ring setting z) which result in case of the construction of an ORV-polar as the synthesis of two known ORV-polars over the two parts L_1 and L_2 of a range of length $L_1 + L_2$

$$\text{(B.1)} \qquad v_{av,s} = \frac{T_1}{T_1+T_2}\,v_{av,1} + \frac{T_2}{T_1+T_2}\,v_{av,2}$$

and

$$\text{(B.2)} \qquad w_{orv,s} = \frac{T_1}{T_1+T_2}\,w_{orv,1} + \frac{T_2}{T_1+T_2}\,w_{orv,2}$$

It was noted thatthese expressions imply that the resulting optimal range velocity vector is a convex combination of the original optimal range velocity vectors and as such in a plot of the ORV-polar (e.g., Fig. 4) will lie on the line connecting the two original optimal range velocity vectors. In addition to that it was noted that the lines through the point (0,z) on the vertical axis and the end points $(v_{av,s},w_{orv,s})$, $(v_{av,1},w_{orv,1})$ and $(v_{av,2},w_{orv,2})$ (corresponding to the same value of z) cut any vertical line in the ORV-polar plot into two pieces, the lengths of which relate to each other following the same ratio as the lengths of the paths L_1 and L_2 of the range. This property of the geometry of the ORV-polar construction will be proved in this appendix with a simple proof from planar geometry (which was supplied by Dr. D. Kijne of the TH Eindhoven).

A geometric picture of the property to be shown is presented in Fig. B.1. It may be noted that it will suffice to show that the lengths of the line pieces AE and ED satisfy the relation

$$\frac{|AE|}{|ED|} = \frac{L_2}{L_1}$$

Given the formulas (B.1) and (B.2) it follows that the lengths of the line pieces AS and SB satisfy

$$\frac{|AS|}{|SB|} = \frac{T_2}{T_1}$$

If in Fig. B.1 the auxiliary line BF is drawn parallel to SE then it follows that

$$\frac{|AE|}{|EF|} = \frac{|AS|}{|SB|} = \frac{T_2}{T_1}$$

as well as

$$\frac{|EF|}{|ED|} = \frac{|CB|}{|CD|} = \frac{v_{av,2}}{v_{av,1}}$$

Combination of these two ratios immediately yields the desired result

$$\frac{AE}{ED} = \frac{AE}{EF} \cdot \frac{EF}{ED} = \frac{T_2}{T_1} \cdot \frac{v_{av2}}{v_{av,1}} = \frac{L_2}{L_1}$$

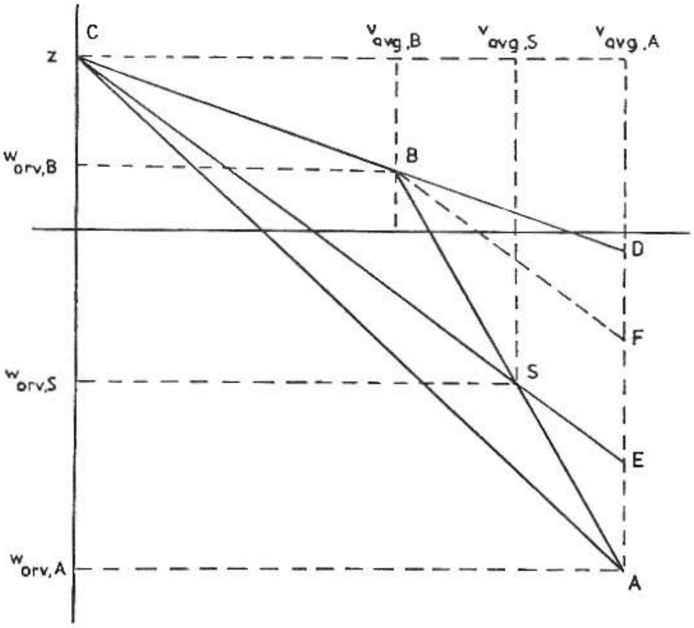

Fig. B.1: Geometric property of ORV-polar construction by syntheses of two known ORV-polars over subsequent ranges

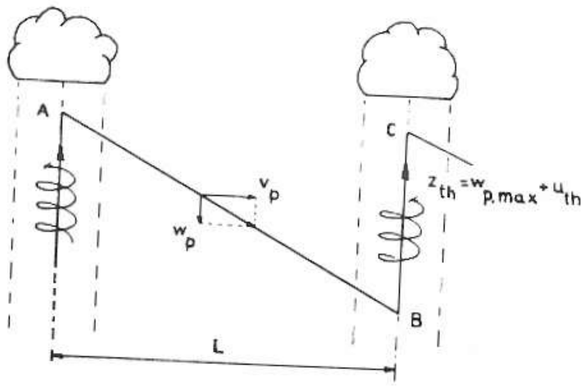

Fig. 1 The classical McCready-problem

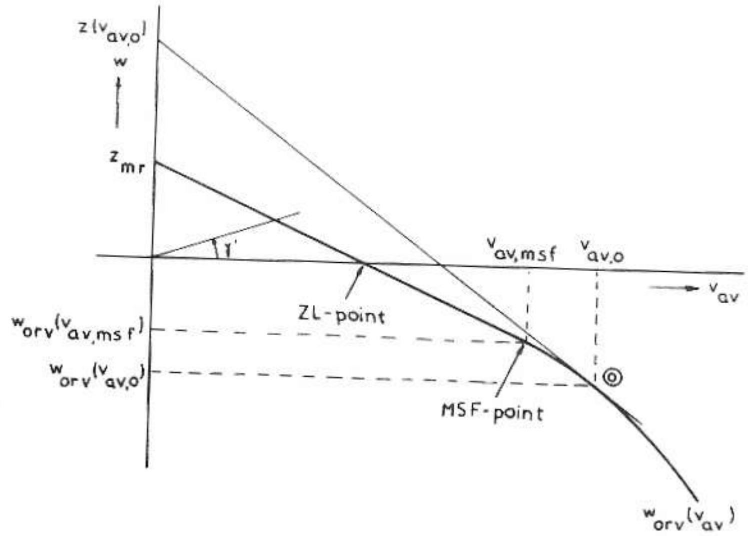

Fig. 3b The ORV-polar for the range in Fig. 3a

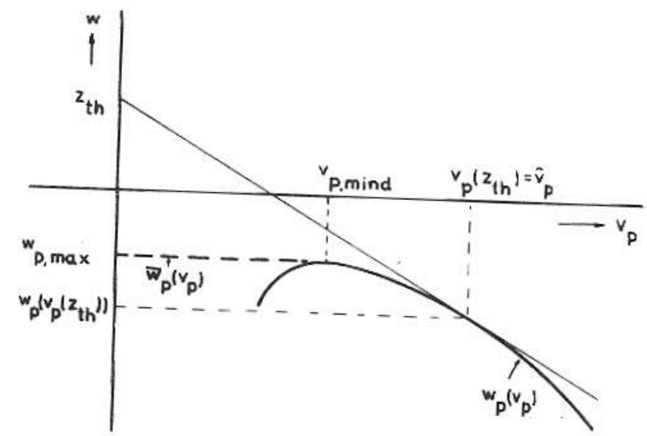

Fig. 2 The (extended) velocity polar of a sailplane & the graphical constuction of the solution of the McCready problem

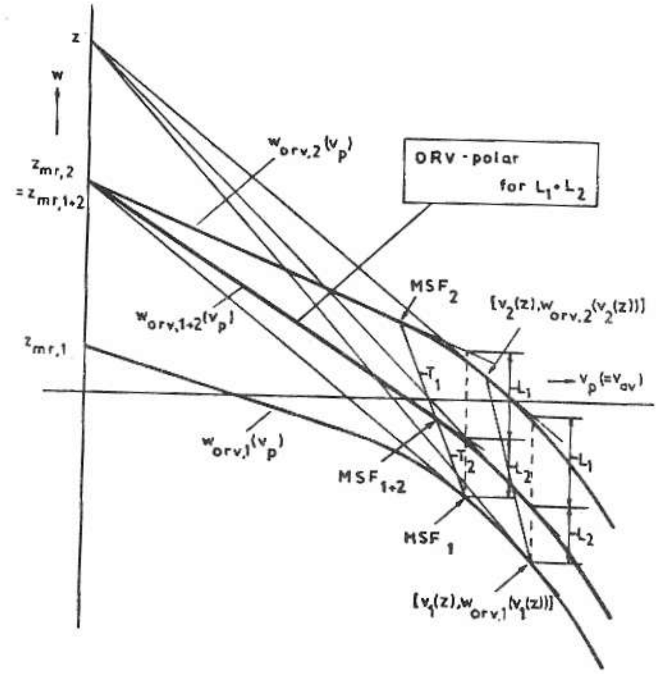

Fig. 4 Sketch of the construction of an ORV-polar by synthesis of 2 known ORV-polars over subsequent ranges

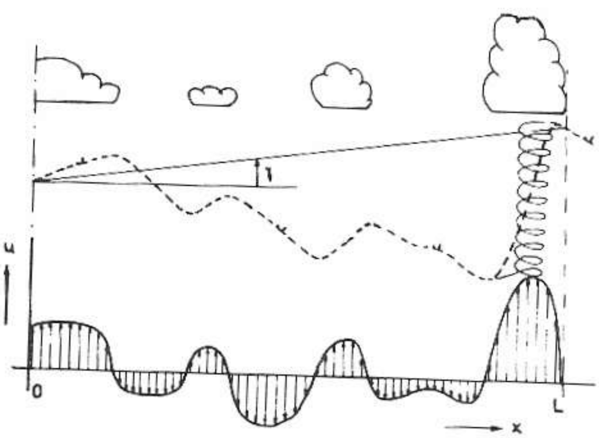

Fig. 3a The generalized dolphin soaring problem

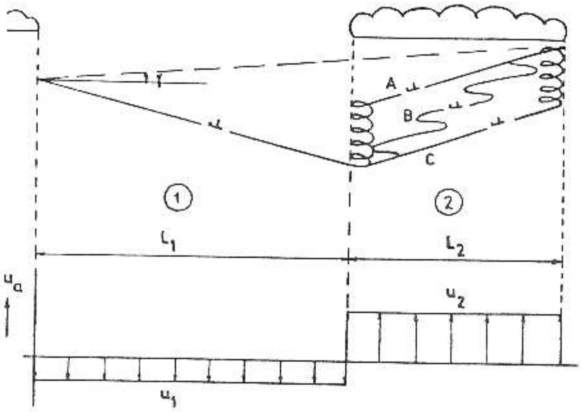

Fig. 5 Sketch of the dolphin soaring problem in case of a square wave thermal model

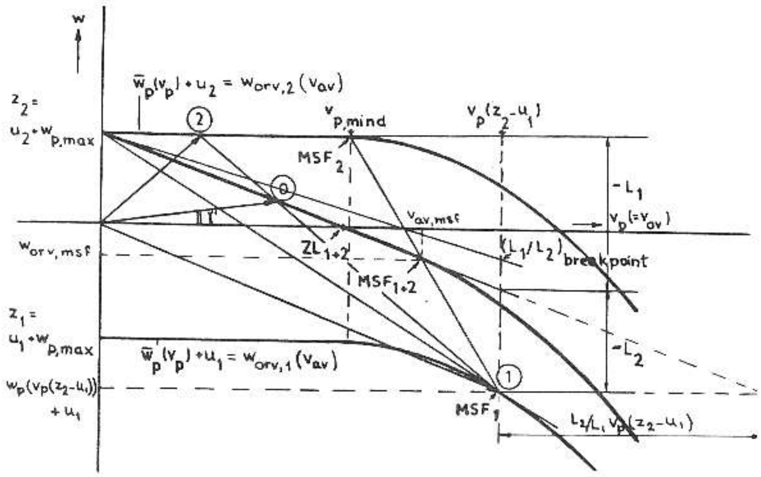

Fig. 6 Graphical construction of the relevant optimal average (horizontal) velocities in case of a square-wave thermal model

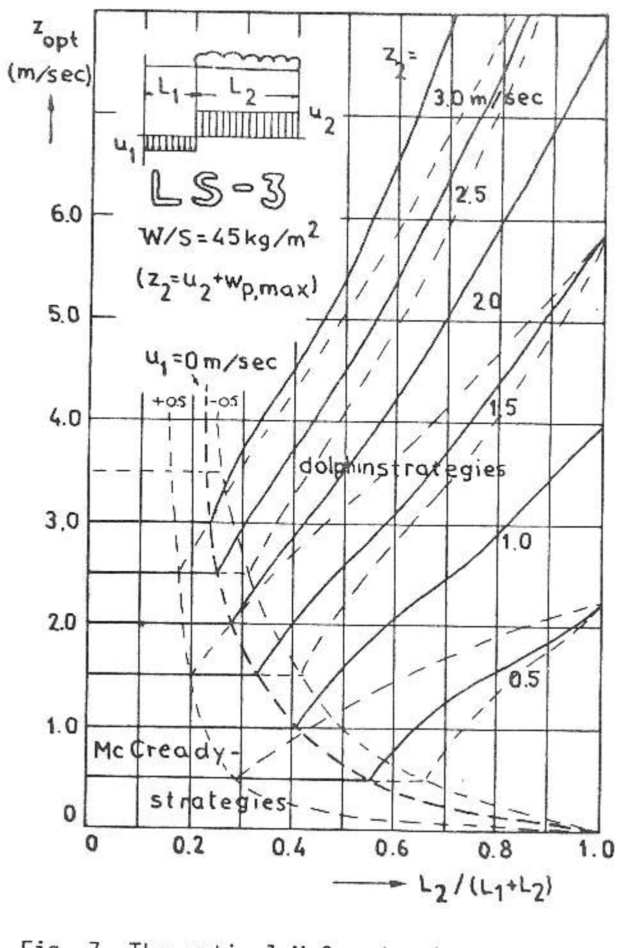

Fig. 7 The optimal McCready-ring settings as a function of the cloud street extension factor for an LS-3 sailplane with wing loading of 33 kg/m^2

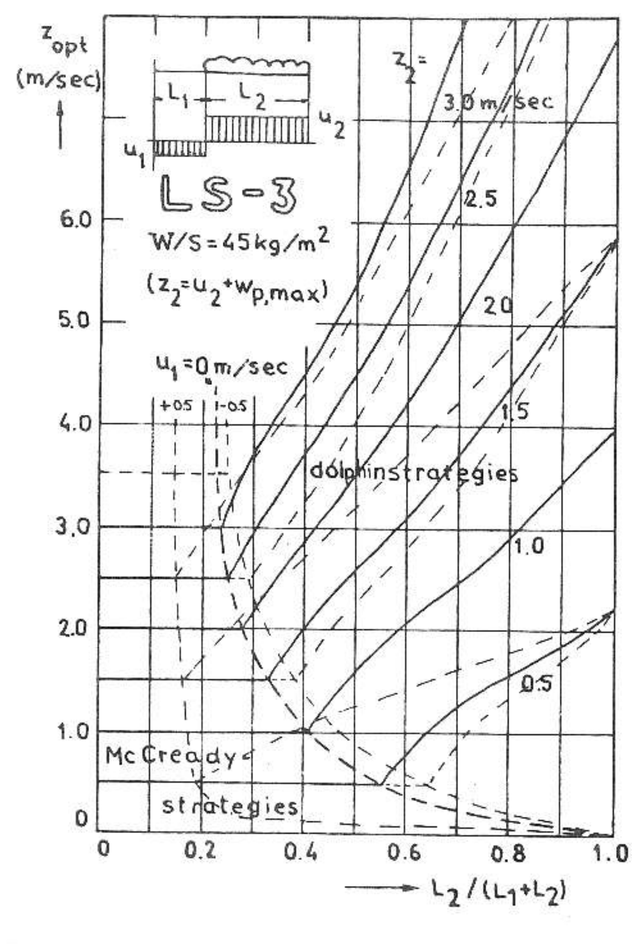

Fig. 8 The optimal McCready-ring settings as a function of the cloud street extension factor for an LS-3 sailplane with wing loading of 45 kg/m^2