STOCHASTIC MODELS OF THERMAL CONVECTION: AN EXTENDED MCCREADY THEORY AND A SIMULATION TOOL

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Abstract

Classical McCready theory is based on a deterministic model. Due to uncertain information during a cross country flight this seems to be rather questionable. The present paper deals with stochastic models which are more appropriate to describe realistic convectional patterns. In a first step the classical theory is extended to the case that the intensity of the next used lift can be characterized by a random variable A. The optimal speed ring setting leading to minimal expected time then is $1/E(1/A)$, where E denotes the expectation. As a full stochastic model, including both random intensities of ups and downs and their extension, we suggest a Markov jump process. An optimal strategy is unknown for this model. However, a simulation tool has been developed to compare the efficiency of different tactics under probabilistic convection profiles of the above type.

I. Introduction

McCready's theory was one of the starting points in the development of analytical models to support tactics of glider pilots. However, this theory is based on a rather simple deterministic model. We feel that a stochastic model is more appropriate, and closer to reality, since the information about lift intensities and distances between lift during a cross country flight is uncertain. In this paper we introduce models of increasing complexity to describe the random nature of convectional patterns. The starting point is an extension of the classical assumption of constant lift intensity. This admits an explicit solution, easy enough to be implemented in forthcoming electronic variometer software. As a more advanced model a Markov jump process is suggested. We cannot offer a full analytical solution, but instead introduce a simulation tool to compare the efficiency of different gliding strategies.

Stochastic models concerning the distance between subsequent lift, and the risc of an outlanding vs. the speed of the glider have already been considered in [1]. We thank Martin Simons for having brought our attention to this nice paper.

For referencing purposes we set out to describe briefly the basics of McCready's theory. A glider is flying at a certain altitude h (=1000 m, say), there is a lift of intensity a at a certain distance d, and there are no up- or downwinds in between. Without loss of generality we may normalize d = 1 (otherwise d would cancel out in subsequent ratios). The question is, which speed has to be chosen by a pilot in order to reach altitude h again

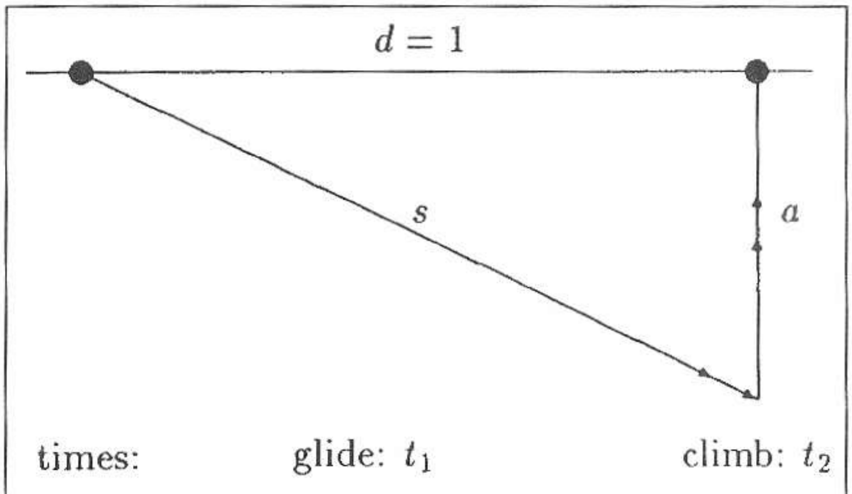


FIGURE 1. The glide and climb path.

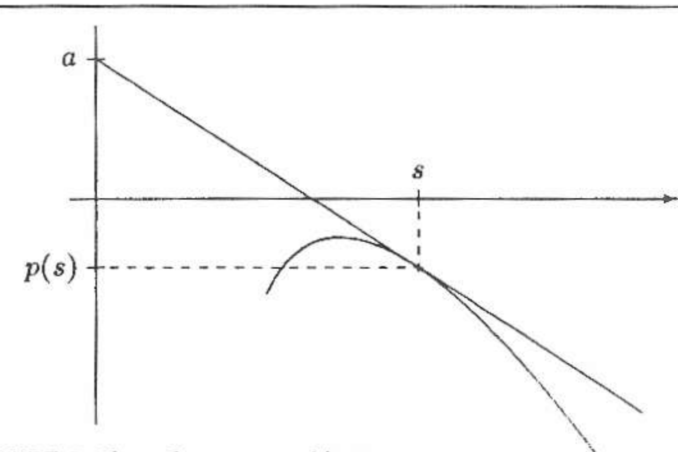


FIGURE 2. The polar curve and its tangent.

after minimal time. He needs time t1 to glide and time t2 to climb back to his original altitude (see Fig. 1). On the other hand, we must consider the performance of the glider, represented by its polar curve *p(s)* (see Fig. 2). When the glider is flown at speed *s*, this results in a sink rate of *p(s)* (in m/s). The following equations now lead to the relevant optimization problem.

$$t_1 = \frac{1}{s}$$

$$t_1 \cdot p(s) = \frac{p(s)}{s} = t_2 \cdot a$$

The first equation is obvious, the second one expresses the fact that the total height lost during gliding equals the height to be gained when climbing. Thus, the total time is

$$t = t_1 + t_2 = \frac{1}{s} + \frac{p(s)}{s \cdot a} = \frac{a + p(s)}{s \cdot a} \,. \qquad (1)$$

Since *a is* a constant, it may be canceled in the numerator above without altering the minimizing speed. Hence, we end up with the following problem

$$\text{minimize } \frac{a + p(s)}{s} \quad \text{over the speed } s \,. \qquad (2)$$

A solution of this problem is easily derived from the polar curve. Find the tangential point at *p(s)* for the line originating from (0, *a*) (see Figure 2). If the corresponding speed is applied during the glide phase, this results in the minimal amount of time to cover distance *d* and climb back to the original altitude. These are the basic ingredients of classical McCready theory; equation (1) allows for an extension to a stochastic model as is derived below.

II. Random Lift Intensity

Usually a pilot cannot be sure about the intensity of the next lift he uses to climb. This uncertain knowledge may be modeled by a random variable A, describing the amount of upwind of the next lift. For instance, a discrete probability model of the following type could be suitable for a certain thermal pattern. The probability that the next used lift

has intensity	0.5	1.0	1.5	2.0	2.5	3.0	m/s
is	0.05	0.2	0.3	0.3	0.1	0.05	

(3)

A continuous model based on the exponential distribution with probability density

$$f_\lambda(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}, \quad \lambda > 0 \text{ a parameter}. \qquad (4)$$

accordingly describes the random lift intensity. In this case the probability that the climb rate is a value between *c* and *d* (m/s) can be calculated as

$$P(c \leq A \leq d) = \int_c^d \lambda e^{-\lambda x} dx, \quad \text{for all } 0 \leq c \leq d.$$

Let E(A) denote the expected value of A. By the strong law of large numbers this is the value one would observe as the arithmetic mean over many measurements of independent lift intensities following the same random law. For distribution (3) the expectation is given by

$$E(A) = 0.05 \cdot 0.5 + 0.2 \cdot 1.0 + 0.3 \cdot 1.5 + 0.3 \cdot 2.0 + 0.1 \cdot 2.5 + 0.05 \cdot 3.0 = 1.675, \qquad (5)$$

whilst in case of (4) we have

$$E(A) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

Revisiting equation (1) under the present stochastic model, the optimization problem reads as to minimize

$$T = \frac{A + p(s)}{s \cdot A} = \frac{1}{s} + \frac{p(s)}{s} \cdot \frac{1}{A} \,.$$

The total time T is now a random variable itself, which cannot be minimized uniformly.

Instead we minimize its expected value, yielding

$$\min_s \mathrm{E}(T) = \min_s \left(\frac{1}{s} + \frac{p(s)}{s} \cdot \mathrm{E}(\frac{1}{A})\right), \qquad (6)$$

because of linearity of the expectation. Abbreviating $b = \mathrm{E}(1/A)$ we get from (6) via

$$\min_s \frac{1 + b \cdot p(s)}{s} = \min_s \frac{1/b + p(s)}{s \cdot (1/b)},$$

after cancelling the constant $1/b$ in the numerator, the problem

$$\min_s \frac{1/b + p(s)}{s}. \qquad (7)$$

The solution is determined in exactly the same manner as above. Find the tangential point at the polar curve for the line now originating from $(0, 1/b)$ (instead of $(0, a)$) . In summary we have the following

Rule. *If the intensity of the next used lift can be described by a random variable A, then the optimum speed ring setting is* $(\mathrm{E}(1/A))^{-1}$, *where 'optimum' refers to minimum expected time.*

This contradicts common opinion that the average of the lift itself should be used as the speed ring setting. The following examples clarify the difference.

Example 1. Assume the discrete distribution given in (3). The corresponding optimal speed ring setting is

$$(0.05/0.5 + 0.2/1.0 + 0.3/1.5 + 0.3/2.0 + 0.1/2.5 + 0.05/3.0)^{-1} = 1/0.7067 = 1.415.$$

This is the so called harmonic mean, yielding a smaller value than $\mathrm{E}(A) = 1.675$. This is generally true for any distribution, and means that the probability of finding better lift than expected does not compensate the worse ones one has to take.

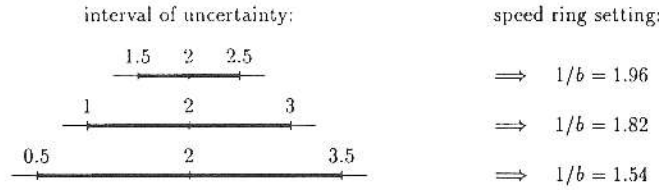
Example 2. Let A be uniformly distributed over $[c, d]$, $0 \leq c < d$, with density

$$f(x) = \begin{cases} 1/(d-c), & \text{if } c \leq x \leq d \\ 0, & \text{otherwise} \end{cases}, \quad 0 \leq c < d \quad \text{parameters.}$$

Roughly spoken, under this distribution any lift intensity between c and d is equally likely. Then $\mathrm{E}(A) = (c + d)/2$, whilst

$$\frac{1}{\mathrm{E}(1/A)} = \left(\int_c^d \frac{1}{d-c} \cdot \frac{1}{x} dx\right)^{-1} = \frac{d-c}{\ln d - \ln c}.$$

The longer the interval $[c, d]$ is, the larger is the uncertainty about the lift coming ahead. The following numerical examples show how to choose corresponding speed ring settings. Observe that in all cases the expected value is the same $\mathrm{E}(A) = 2.0$.



Taking limits $c,d \rightarrow 2$ yields $1/b = 2$ such that the deterministic model is obtained as a special case.

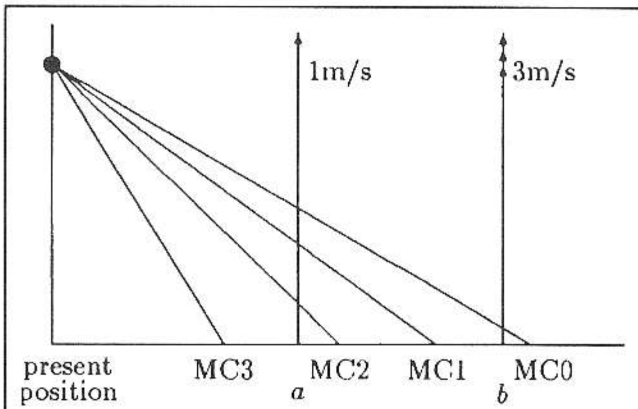


FIGURE 3. Glide path and lift.

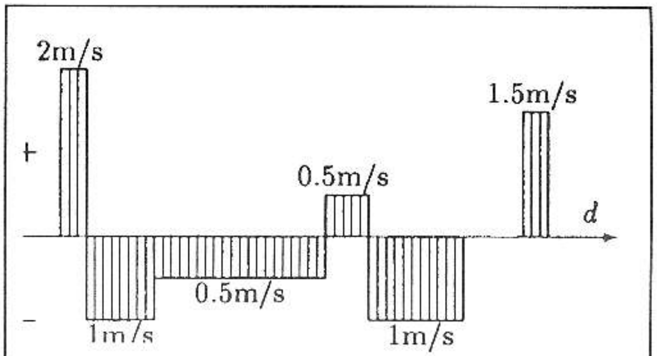


FIGURE 4. Ups and downs on a track.

III. A Full Stochastic Model

Things are even more complicated during a real flight. Not only the (random) intensity of lift, but also their distance essentially influences the optimum strategy. The following small example reveals the additional difficulties. Consider a glider at 1000 m above ground, furthermore a lift of 1 m/s at distance a and another one of 3 m/s at distance b. Glidepaths corresponding to speed ring settings 0,1,2, and 3 are depicted in Figure 3. The corresponding points where the glider would touch the ground without climbing on its way are denoted accordingly by MC0, ..., MC3. Dependent on the position of the stronger lift the time minimal strategy to climb back to the original altitude in this lift is the following.

MC1 $\leq b$: Go to the lift at position a with speed ring setting 1. Climb to the altitude to just reach the lift at b

using speed ring setting 1. Climb in the lift at b.

MC3 $\leq b \leq$ MC1: Go to the lift at b using speed ring setting x, x as large as to just reach the lift at b. Forget about the lift at a. Climb in the lift at b. $b \leq$ MC3: Go to the lift at b using speed ring setting 3. Climb there.

Now, if the intensity of lift b is a random variable and the position is fixed, this would result in a complicated optimal-on-average strategy with many subcases to be distinguished. If the distance itself is a random variable, then characterizing the optimal strategy in a concise manner seems to be hopeless. Moreover, to describe random convection patterns over a long distance in this way leads to an untractable model.

First of all, a reasonable but still tractable model of convection behaviour is needed. We suggest to observe convection only along the path of a glider projected onto the ground, and record ups and downs along this path. This of course means that pilots flying along different paths in the same weather conditions experience different convection profiles. But this is natural, competing pilots try to follow those tracks which promise most favourable conditions. Our intention is to determine the optimum speed once the track has been chosen. After discretization (e.g. in steps of 0.5 m/s from -3 m/s to +3 m/s) one could for example observe the contour represented in Fig. 4. There are a lot of empirical rules for typical thermal patterns. For instance, 'The stronger an up- or downwind is, the smaller is its average extension.', or 'To have a strong down- after a strong upwind is most likely'. Many other empirical rule exist. We add one more, which seems not to contradict reality: 'The future convection on a path depends only on the present, not on past values of ups and downs'.

Markov jump processes are able to model these empirical rules. A Markov process is a family of random variables $X(d)$, $d \geq 0$, where $X(d)$ denotes the random value of convection at distance d from the origin on a discrete scale S, e.g. as above $S = \{-3.0, -2.5, \ldots, 0, \ldots, 2.5, 3.0\}$. A concise description of the underlying theory can be found in [2]. The stochastic behaviour of the process is completely described by the initial distribution and the corresponding generator matrix $Q = (q_{ij})_{ij \in S}$, with $q_{ij} \geq 0$, if $i \neq j$, and $q_{ii} = -\sum_{j \neq i} q_{ij}$. Roughly spoken, $p_{ij} = q_{ij} / -q_{ii}$ is the probability to visit convection of intensity j when leaving state i. Furthermore, the sojourn time to stay in state $j \in S$ is exponentially distributed with parameter $-q_{jj}$.

Example 3. The following very simple model considers only ups and downs on the scale

$$Q = \begin{pmatrix} -4 & 3 & 1 \\ 3 & -3 & 0 \\ 5 & 1 & -6 \end{pmatrix}.$$

Then the probability for a jump from -1 m/s to +1 m/s is 1/4, and from - 1 m/s to 0 m/s is 3/4. The extension of a +1 m/s lift is an exponentially distributed random variable with expectation 1/6. The distance measuring unit in this case could be 100 m, such that lift of +1 m/s have an expected extension of 100 m.

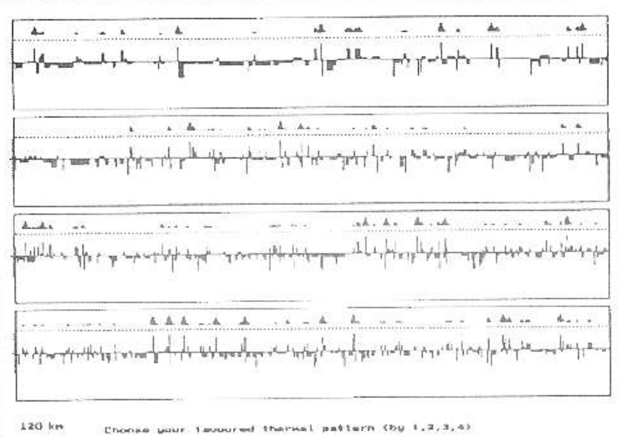


FIGURE 5. Thermal patterns.

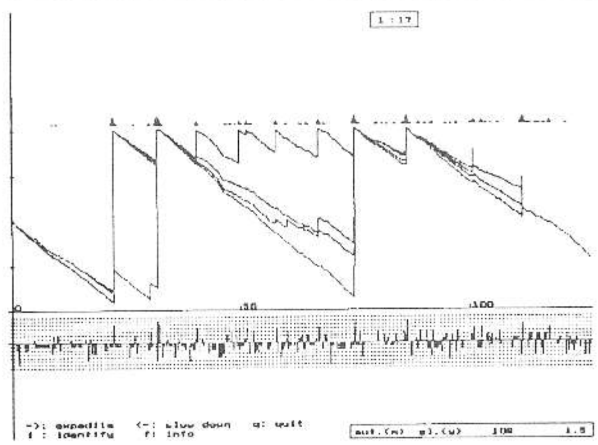


FIGURE 6. Sample screen of glidepaths.

The advantage of the Markov model is its dependence on only a few parameters. If the state space S contains n elements (up/down values), then the model is specified by $n(n - 1)$ parameters. These offer great flexibilty, while still retaining tractability of the model.

The optimal strategy to cover a certain distance in minimum expected time is unknown under this model. However, to compare the efficiency of different strategies, a simulation tool has been developed. The simulation process consists of two steps. First, a weather pattern is offered according to the above model. Corresponding parameters in form of an intensity matrix can be fed by the user. Furthermore, one out of four different factors can be chosen, causing more rapid changes in the same basic convection pattern. Lifts are visualized by clouds and the corresponding intensity can be recognized from a scale below (see Figure 5). After the convection pattern has been generated, the gliding paths of four gliders are depicted, each applying a different strategy (see Figure 6 for a sample screen). The gliders use the optimal speed according to individual tactics of variable speed ring setting and the present convection intensity. The actual sink rate is then taken from the polar curve such that a realistic performance behaviour is achieved.

One of the gliders can be controlled manually such that a pilot can practice his own tactics in a random weather environment, and compare it to other competitors. Several options can be selected by the user, namely convection parameters (see above), ceiling, distance of the task, visibility (which means the distance to which lift can be seen ahead of the present position of the manually controlled glider), and the type of active gliders with or without water ballast.

Extensive use of the simulator has shown that it generates quite realistic convection patterns and allows for comparing strategies in an actual environment. It thus can be used as a training program to improve a pilots tactical decision making. The corresponding program runs on PCs with VGA graphics card. It will be made available by the author upon request.

IV. Conclusions

In section II we suggested an extended McCready theory based on random lift intensities. The corresponding optimization problem has a simple solution based on a modified speed ring setting. By estimating the distribution of the lift intensities from past data during a task, the corresponding setting could be calculated automatically by a modern electronic variometer, thus supporting the decision of pilots on their speed to be flown between successive lift. Concerning the Markov model, a lot of questions is still open. Of paramount interest is the problem how to fit the parameters of the model to existing measured data of convection patterns. This would verify the usefulness of the model for generating realistic random environment. Further work will also be devoted to determine good, maybe nearly optimal strategies which simultaneously minimize the expected time en route and the risc of an outlanding.

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