OPTIMIZING FIGURE-8 SOARING FOR HILL AND WAVE LIFT

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1. INTRODUCTION.

Flying to and fro along a line of hill or wave lift we often pass through local patches of stronger lift. Traffic and other conditions permitting, we sometimes try to work this lift either by circling, by flying short beats through the good patch, or by doing figures of eight. Unlike thermals, which drift downwind, hill and wave lift tend to be associated with the ground features producing them. Shifts certainly occur, but there are periods when the area of lift seems to be fixed near some point on the ground, and we try to stay fairly close to it. This note suggests ways to optimize the gain from the figure-8 method.

In what follows we will take the direction from which the wind blows as 0° (≡ 360°) and the line of lift as running across wind (090° ↔ 270°). We will first consider a simple case, consisting of a sequence of reverse turns.

2. SIMPLE CASE: 180° TURNS, TO AND FRO ACROSS WIND.

Suppose we start from a heading directly across wind, say 090°. First turn left (upwind) onto 270°, then roll into the opposite turn back onto 090°, and so on. **Relative to the air**, if we ignore the time taken in rolling from one turn to the other, the result is a "chain" of semicircles (see Figure 1). Let the radius of the turning circle be **r**, and **c** the circling time, for the chosen airspeed and angle of bank.

Let **A** be the airspeed, **W** the windspeed, and **R = W/A**. In this section we show that for a certain fixed value of **R**, the path of the glider **relative to the ground** is a figure of eight centered at a fixed point on the ground, above which the turn reversal occurs.

Relative to the air (see Figure 1) the glider in completing the semicircle SXN moves a distance **2r** in the direction SN. This takes time **c/2**. **Relative to the ground** the airmass moves a distance **Wc/2** in the direction NS. Let **s** be the distance in the direction SN that the glider moves. Using $\mathbf{Wc/2 = \pi rW/A}$ we get

$$\mathbf{s = 2r\,(1 - R\pi/2)} \qquad (1)$$

We see that $\mathbf{s = 0}$ when $\mathbf{R = 2/\pi = 0.6366}$.

In this case the net result of completing a semicircle is zero displacement in the upwind or downwind direction. By symmetry the displacement perpendicular to NS (across wind) after each complete semicircle is zero. Thus each turn reversal takes place over the same fixed

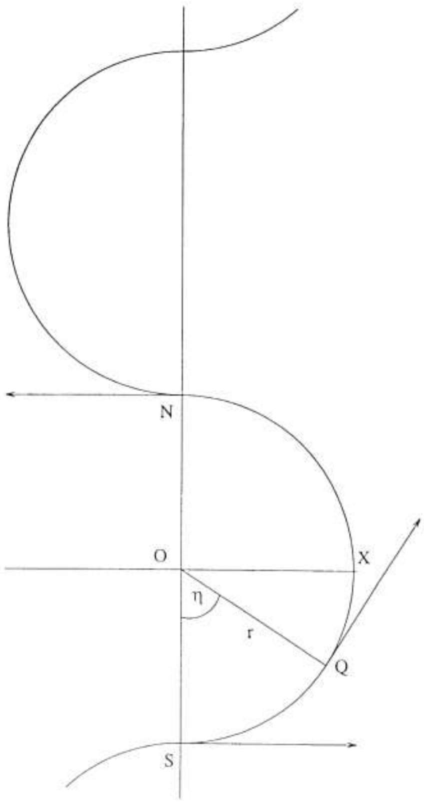


FIGURE 1. "Simple Case" of Sections 2 and 4. "Chain" of semicircles forming S-turns either side of wind direction.

point on the ground.

Numerical example: **$s = 0$ when $A = 45$ kts and $W =$ 28. 64 kts.**

3. OUTLINE OF THE REST OF THE PAPER

There are nine numbered sections. In some, the main text is followed by some paragraphs of remarks and consequences: these are numbered separately as, for example (4.i), (4.ii) etc.

Section 4: Detailed examination of the path relative to the ground, showing that the method keeps the glider within a small area, which is likely to increase the time spent in stronger lift.

Section 5: Development of the method of flying a figure-8, fixed relative to the ground, for any value of **R** **($0 \le R \le 1$)**.

Section 6: Detailed examination of the paths in section 5.

Section 7: Technique for flying the figure-8.

Section 8: Proposals for evaluating the effectiveness of the method.

Section 9: Some other applications to low flying and safety problems (also applicable to power flying).

4. SIMPLE CASE: PATH RELATIVE TO THE GROUND.

In Figure 2, suppose the semicircle SXN is the path **relative to the air**. Suppose the glider flies round the arc SQ where the angle $S\hat{O}Q = \eta$ (radians) and $0 < \eta < \pi/2$.

This takes time $r\eta/A$, during which the air mass moves a distance $Wr\eta/A = rR\eta$ in the direction NS.

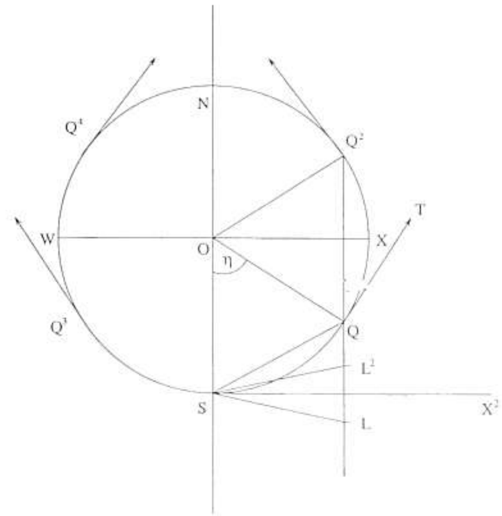


FIGURE 2. Vector diagram for equation (2).

Let **L** be the position of the glider **relative to the ground** due to flying round the arc SQ **and** to the movement of the air. At the starting time, the point S represents the position of the glider **both relative to the air and the ground**, so that the vector $\overline{SQ}$ is the **displacement relative to the air**, and **SL** is the sum of this and the **displacement relative to the ground.** By protecting vectors on the axes $\overline{OX}$, $\overline{ON}$ and using $R = 2/\pi$ this gives:-

$$\overline{SL} = \overline{SQ} + \overline{QL}$$

$$= \overline{OQ} + \overline{QL} - \overline{OS}$$

$$= r\,(\sin\eta\ -\cos\eta) + r\,(0, -R\eta) - r\,(0, -1)$$

$$= r\,(\sin\eta, 1 - \cos\eta - 2\eta/\pi) \qquad (2)$$

Calculation of values is elementary. The graph of the results for $0 \le \eta \le \pi/2$ is the lower boundary of the right-hand lobe of the figure-8 in Figure 3. In Figure 2 the direction of the tangent QT is the heading when the glider is at Q (relative to the air) and at L (relative to the ground). From the shape of the lobe, drift and track are non-linear functions of η.

(4.i) Completion of the figure-8 relative to the ground.

The following considerations of symmetry show that no further calculations are required. See Figure 2 where superscripts indicate points and angles related symmetrically to Q and L. Consider Q^2 where $S\hat{O}Q^2 = \eta^2 = \pi - \eta$ so that $\pi/2 < \eta^2 < \pi$ and let $\overline{SL}^2 = r\,(\sin\eta^2, 1 - \cos\eta^2 - R\eta^2$.

Then,

$$\overline{SL}^2 = r\,(\sin\eta, 1 + \cos\eta - R\pi + R\eta)$$

$$= r\,(\sin\eta, -1 + \cos\eta + 2\eta/\pi) \qquad (3)$$

From (2) and (3), L and L^2 are symmetrically placed with respect to the axis SX^2. It follows that as the glider flies round the semicircle SXN relative to the air, its path over the ground is anticlockwise round the right-hand lobe in Figure (3). By symmetry about the axis SN (note Q^3 and Q^4) we see that a flight upwind around the left semicircle SWN results in the clockwise path around the left hand lobe.

To graph the complete figure-8 it suffices to plot the four points

$$(\mathbf{L}, \mathbf{L}^2, \mathbf{L}^3, \mathbf{L}^4) = \mathbf{r}\,(\pm \sin \eta, \pm \{ 1 - \cos \eta - 2\,\eta/\pi \}) \qquad (4)$$

We see that the diameter of the figure-8 perpendicu-

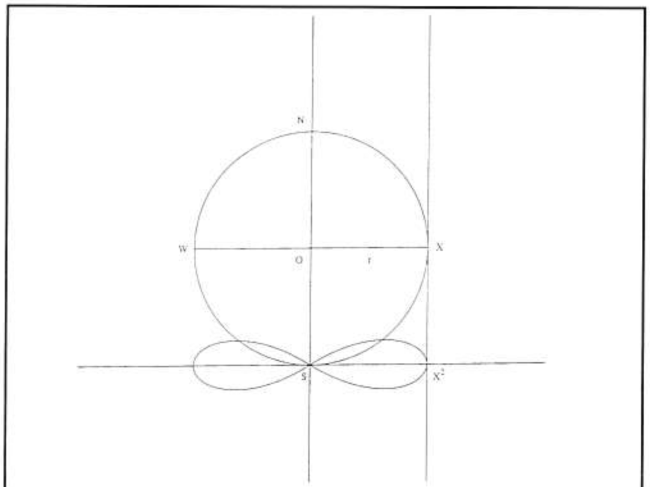


FIGURE 3. "Simple Case". The figure-8 is shown to scale which is fixed by the radius r of the turning circle.

lar to $\overline{SN}$ is 2r (the diameter of the turning circle). For the diameter parallel to $\overline{SN}$ we need only find the minimum of $\mathbf{1 - \cos \eta - 2\,\eta / \pi}$ in $\mathbf{0 < \eta < \pi}$. The value of η for this is $\mathbf{\sin^{-1} (2/\pi)}$. The position vector of the minimum point is

$$\overline{\mathbf{SL}}_{\mathbf{min}} = \mathbf{r}\,(2/\pi, 1 - \sqrt{1 - \{2/\pi\}^2} - \{ 2/\pi\} \sin^{-1} \{2/\pi\}) \qquad (5)$$

$$= \mathbf{r}\,(0.6366, -0.2105)$$

which shows that the diameter of the lobe parallel to $\overline{SN}$ is **0.421r**. Compared to the turning circle shown (of diameter 2r) the glider moves over a much smaller range. Also it is likely to be an advantage that its largest excursions will be approximately along the line of lift.

(4.ii) Remark on Scales.

In the above calculation, **r** η is the length of arc SQ and clearly η must be in radians. However the argument in **sin** η may be in radians or degrees depending on the program or tables being used. For use in the air, the final results will be true or magnetic headings, and these will be indicated as being in degrees. Otherwise we will only indicate degrees or radians where there might be uncertainty.

The turning radius **r** is a scale constant depending on the airspeed **A** and the angle of bank. In tabulating results we will take **r = 1**. Results for specific numerical cases can easily be derived. Further optimization may be possible by alterations in **A** or the angle of bank (and hence **r**) but these are dependent on the aircraft type, and other factors. In the present work we treat them as fixed.

(4. iii) Varying values of R.

Since **R = W/A**, it can take any value in [0, 1] according to wind and choice of airspeed. We see from equation (1) that if the method of 180° turns is adopted when **R** < [>] **2/π** the pattern on the ground will move upwind [downwind].

In sections 5 and 6 we will derive a method to produce a fixed figure-8 pattern for each given **R**. From the above remark we can also see that the pattern can be shifted up- or downwind as required to look for increased lift.

(4. iv) Headings.

Referring to Figures 1 and 2, note that the position on the turning circle at the instants of turn reversal are **S** and **N**, and the corresponding headings are **90°** and **270°**

5. GENERAL CASE FOR ANY R (0 ≤ R ≤ 1): POSITION FOR TURN REVERSAL.

We see from (4. iv) that when **R = 0. 6366** the optimal figure-8 results from reversing the turn on headings of **90°** and **270°**. We now show that for any **R (0 ≤ R ≤ 1)** there is a corresponding angle θ **(0 ≤ θ° ≤ 180°)** such that the optimal technique is to reverse on θ° and **360°** - θ°. This angle (in radians) is given by the equation

$$\mathbf{-\sin \theta - R\theta = 0} \qquad (6)$$

See Figures 4 and 7, where the chains of circular arcs **relative to the air** are shown for **R = 8/9** and **R = 1/3** respectively. In Figures 5, 8 and 9, the turning circle is shown in its instantaneous position at the starting time, and the angles PÔX = XÔP′ = θ. Figures 4 and 5 correspond to **R > 2/π** which gives **0 < θ < π/2** and 7, 8 and 9 to R < 2/π which gives **π/2 < θ < π.**

In flying round the circular arc PXP′, **relative to the air**, the glider will go round a closed curve, **relative to the ground**, starting and finishing at P. To verify this statement, note that P and P′ play the roles of S and N in section 2. By exactly similar arguments we see that the distance **s** moved, **relative to the ground** in the direction SN is

$$\mathbf{s = 2r\,(\sin \theta - R\theta)} \qquad (7)$$

which is zero when condition (6) holds.

(5. i) Remarks on special values of R.

R = 0 corresponds to **W = 0**, and gives **sin** η **= 0** whence (assuming **A > 0**) θ = π. The glider flies round the complete semicircle X′ SX. In fact it would remain in a fixed pattern by continuous circling, as is obvious in the condition of nil wind.

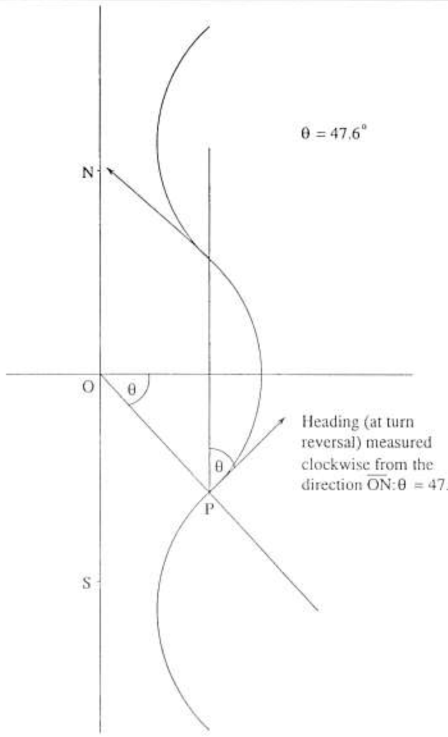


FIGURE 4. For angle θ see section 5, and ratio R (4.iii). Here $R > 2/\pi$ giving $\theta < 90^\circ$. S-turns are reversed when the heading is $\pm\,\theta$ from ON.

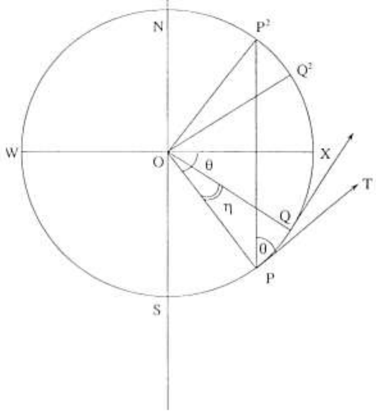


FIGURE 5. Vector diagram for section 6 (case 1). For the variety of cases, see equations (8) through (12).

R = 1 corresponds to **W = A**. The glider can only keep station by heading directly upwind. The equation $\sin\theta = \theta$ has the root $\theta = 0$, and with non-zero wind this is possible (though in practice only when **W** > stalling speed).

$R = 2/\pi$ is the "simple case" of section 2. This can now be seen to be the boundary case between stronger winds in which the glider can keep station by weaving to either side of upwind, and weaker winds in which it is necessary to turn partly downwind in each cycle.

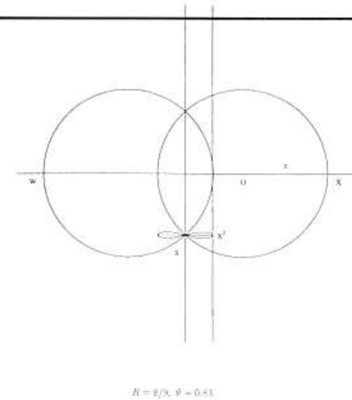


FIGURE 6. Scaled by r as in Figure 4. Conditions reversed: $R < 2/\pi$ giving $\theta > 90^\circ$. Note much smaller figure-8.

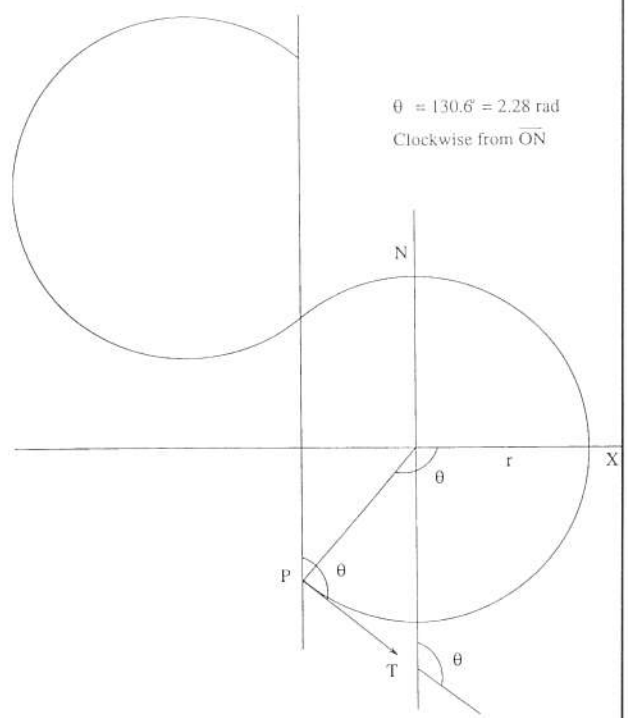


FIGURE 7. S-turn under the conditions of Figure 6. Note how this is looped in contrast to Figures 3 and 5.

6. GENERAL CASE: PATH RELATIVE TO THE GROUND. $(0 \le R \le 1)$

We show that the equation of the vector $\overline{PL}$ as a function of the parameters η and θ is (generalizing (2)):

$$\overline{PL} = r\,(\cos\{\theta - \eta\} - \cos\theta,\ -\sin\{\theta - \eta\} + \sin\theta - R\eta) \qquad (8)$$

The argument at the beginning of section 3 concerning flying round the arc $\overline{SQ}$ gives, for a general starting point P instead of S, the vector equation

$$\overline{PL} = \overline{OQ} + \overline{QL} - \overline{OP} \qquad (9)$$

As in section 4 we project vectors on the axes $\overline{OX}$, $\overline{ON}$. Trigonometrical details depend on the order of the fixed

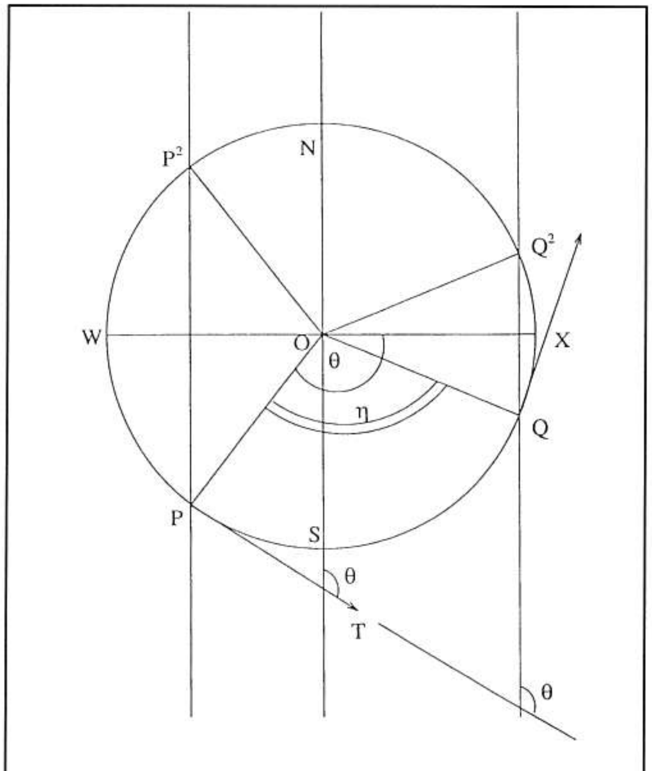


FIGURE 8. Vector diagram for section 8 (case 2).

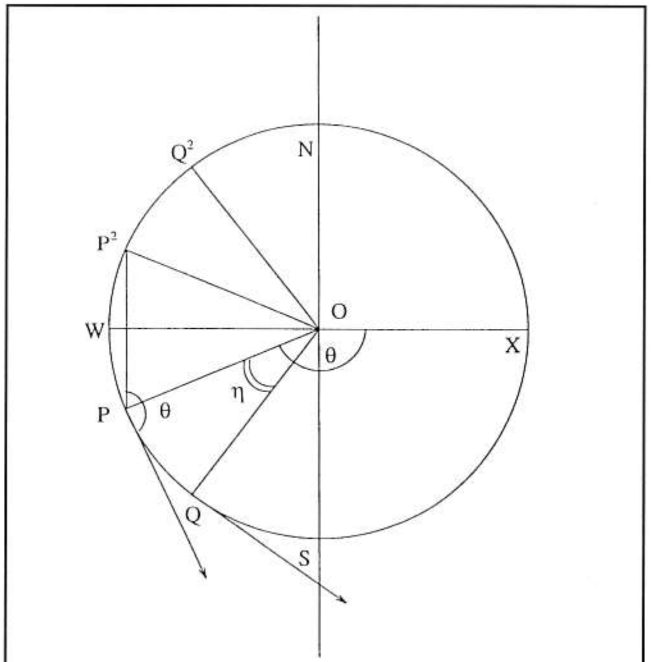


FIGURE 9. Vector diagram for section 8 (case 3).

points (W, S, X) and the variable ones (P, Q). The possibilities are illustrated in three figures as follows: - Figure 5 (W, S, P, Q, X); Figure 7 (W, P, S, Q, X); Figure 8 (W, P, Q, S, X).

For the vector $\overline{QL}$ note that its direction is NS and its magnitude $|\overline{\mathbf{QL}}| = \mathbf{R}\eta$ so that

$$\overline{\mathbf{QL}} = \mathbf{r}\ (\mathbf{0}, -\mathbf{R}\eta\ ) \tag{10}$$

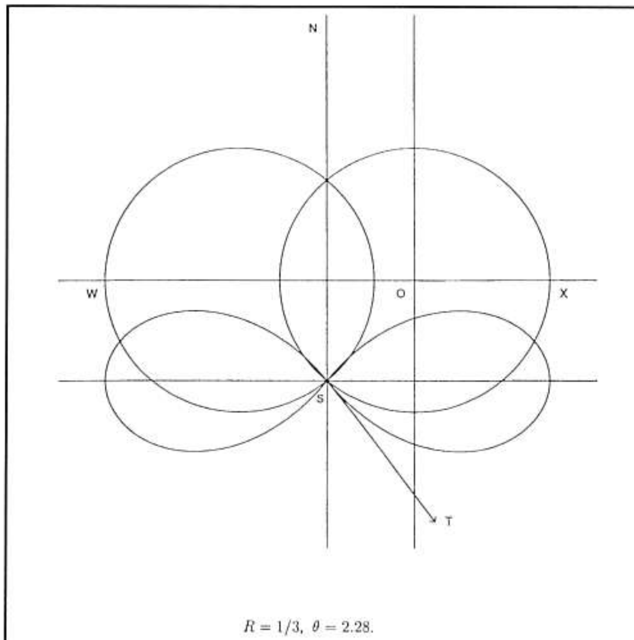


FIGURE 10. With Figure 11, this illustrates the behavior of the figure-8 relative to r as R decreases (weakening wind).

For $\overline{OP}$ there are two cases. In Figure 5, $\mathbf{P\hat{O}X} = \theta$ so projecting on $\overline{SN}$, $\overline{OX}$

$$\overline{\mathbf{OP}} = \mathbf{r}\ (\cos\theta, -\sin\theta\ ) \tag{11}$$

In Figures 8 and 9,

$$\mathbf{P\hat{O}S} = \theta - \pi/2$$

so that

$$\overline{\mathbf{OP}} = \mathbf{r}\ (-\sin\{\theta - \pi/2),\ -\cos\{\theta - \pi/2\})$$

which by simple trigonometrical identities is the same as (11).

For $\overline{OQ}$, in Figures 5 and 8, $Q\hat{O}X = \theta - \eta$ (the heading at Q) so

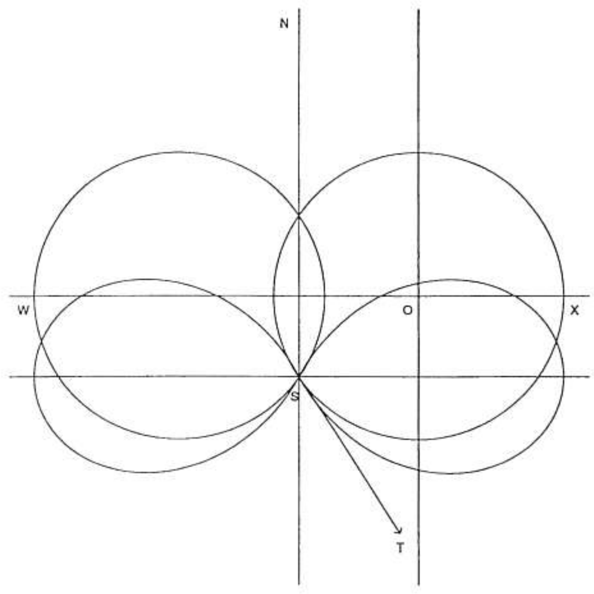
$$\overline{\mathbf{OQ}} = \mathbf{r}\ (\cos\{\theta - \eta),\ \sin\{\theta - \eta\}) \tag{12}$$

In Figure 9 we have $\mathbf{Q\hat{O}S} = \theta - \eta - \pi/2$ and again using identities this gives (12). Combining (9), (10), (11), (12) gives (8) as stated.

(6. i) Shape of the path relative to the ground.

The remarks on symmetry in section 3. (i) carry over almost unchanged, to show that the path relative to the ground is a figure of eight centered at the origin P with axes parallel, and perpendicular, to OX and SN. Examples will be seen in the figures for various values of R in the appendix, plotted using equation 8.

(6. ii) Maximum distances from P in direction of the axes.



$R = 2/9,\ \theta = 2.54.$

FIGURE 11. With R = 1/3 in Figure 10 and R = 2/9 here, note the lobes of the figure-8 expanding to fill the turning circles.

As in section 3, there is a maximum width of the lobes, or diameter, in the direction parallel to SN. This is a function of R, and the diameter of the figure-8 parallel to OX also varies with R. Values of these provide a useful impression of the distances traveled away from the center with varying air- and windspeeds, to complement the plots reproduced.

Differentiating (8) with respect to η gives (13)

$$(d/d\eta)\ \overline{PL} = r\ (\sin\{\theta - \eta\},\ \cos\{\theta - \eta\} - R) \qquad (14)$$

Let η_{min} be the value in $0 < \eta < \pi/2$ for which the (negative) second coordinate is a minimum, given by

$$\cos\{\theta - \eta_{min}\} = R \qquad (15)$$

Substituting from this in (8), and also using equation (6) gives the position of the minimum

$$\overline{PL}_{min} = r\ (R - \sqrt{1 - R^2}\theta^2,\ R\cos^{-1}R - \sqrt{1 - R^2})$$

7. TECHNIQUE FOR FLYING THE OPTIMAL FIGURE-8.

This section, together with sections 8 and 9, will be concerned with work in progress, further proposals, and some general comments. There is no claim of completeness, and some results will be presented in the form of examples.

(7.i) Simple Case: Use of landmarks.

To illustrate the various requirements, and one solution, let us consider the simple cases of sections 1 and 2. Assume airspeed **A = 45** kts. windspeed **W = 28.64** kts. and wind direction 0° (from). Suppose suitable lift is encountered while flying upwind, due North. Then the pilot should turn onto 090°, reverse direction onto 270°, and so on.

The only problem is to perform the first 90° turn, and subsequent 180° ones with sufficient precision. In good visibility this is easy: before the first turn, note landmarks (preferably prominent and distant) or conspicuous clouds off both wingtips. Reversing the turn when heading, alternately, towards one and then the other of these marks results in flying the special figure-8 of the simple case.

(7.ii). General Case for any R (O < R < 1): Further Use of landmarks.

For the above technique to be useful it must clearly be extended to allow for any combination of **A** and **W** that is likely to be encountered. The basis of this has been laid in Sections 5 and 6. Note in particular the role of the solution θ to equation (6). A table of values of θ° will be found in the Appendix at the end of the paper. It is useful to note values of θ° for likely combinations of **A** and **W**, on the basis of met. observations, before flying. See section (4.ii) for the use of degrees here and below.

The use of landmarks allows us to treat the upwind direction as 0°, as in (7.i). Then the problem is to reverse the turn on the alternate headings θ° and **360**° - θ°. This depends on the wind strength:-

(a) In strong winds (**R** near **1**) θ° is small, and looking ahead it is possible to judge a weave of a few degrees either side of the upwind direction.

(b) In weaker winds θ° is larger, but an approximate measured turn can still be based on the 90° angle between aircraft heading and wingtip direction, by scanning the quadrant between nose and wingtip, and dividing it by eye. If θ° < **90**° then a landmark bearing θ° can be approximately located in this way. If θ° > 90° turn through 90° and then a further θ° - **90**°. The opposite reverse turns are, from North, the same magnitudes but to the left.

This procedure of locating landmarks is approximate but simple especially if it is practiced in advance. Useful landmarks can be noted, and judgment of angles improved by doing turns onto headings and then letting the compass settle. Noting landmarks and their bearings can also be done during an actual search for lift.

(7. iii) Use of Instruments.

The compass could obviously take the primary role in fixing the turn reversals, and for these it is necessary to allow for the wind direction. Let the wind direction be δ°. In (7.i and ii) δ° = **0**° and the headings for turn reversals are + θ° (when turning right) and - θ° (left).

Changing the wind direction to δ° changes these to δ° + θ° and δ° - θ° respectively. Note that the first of these may exceed 360° while the latter may be negative. Conventionally a compass heading will be within the interval $(0°, 360°)$. Let the compass headings on which to reverse from a right [resp. left] turn be $\alpha°$ [resp. $\beta°$]. Using the fact that $\mathbf{0 \leq \delta° \leq 360°}$ and $\mathbf{0 \leq \theta° \leq 180°}$, we find that the conditions are satisfied by the following special cases of addition and subtraction modulo 360:-

$$\alpha° = \begin{cases} \delta° + \theta° \text{ if this } \leq 360° \\ \delta° + \theta° - 360° \text{ otherwise.} \end{cases}$$

$$\beta° = \begin{cases} \delta° - \theta° \text{ if this } \leq 0° \\ \delta° - \theta° + 360° \text{ otherwise.} \end{cases}$$

Glider pilots who have practised the art of turning onto headings on a magnetic compass will easily develop the technique. At altitude in wave the turns can often be fairly gentle, which may make it easier. Lower down, and particularly when trying to make the transition from hill soaring to wave, tighter turns and higher airspeeds are likely to be necessary. A directional gyro would help. When there is no horizon, instruments may be necessary for the turn reversals, even though the glider is clear of cloud.

When searching for lift on any heading in good visibility, it is useful to be prepared with the appropriate values of $\alpha°$ and $\beta°$. With this, one can start a turn in the preferred direction as soon as good lift is encountered, and reverse on the correct heading by compass. A flexible approach, with cross reference between landmarks and compass, seems generally desirable.

8. PROPOSALS FOR EVALUATION OF THE METHOD.

No systematic attempt to evaluate the technique has yet been possible, but on the relatively few flights on which it has been tried there have been some encouraging results. These were not evaluated against defined criteria, and, in the small sample, variation of conditions such as weather, presence of other gliders to compare performance, etc. meant that no two flights could be precisely compared. Nevertheless, observations such as climbing above gliders of lower wingload, or being the only one of a group to climb from hill lift into wave seemed, as just said "encouraging."

An obvious proposal is that more people should try the technique. If they find it produces markedly better performance than they had expected in the conditions, the term "encouraging" would be reinforced.

In the longer run, however, a systematic evaluation will be needed. After the first successful climb, the Author (a statistician) sobered up on the thought that anything that kept the pilot more or less in the right area, and helped to concentrate the mind and hands, would produce the same result when a puff of better lift came along.

It is not only in the case of apparent success that a test would be useful. If the technique increased the proportion of winch launches into hill lift that led on to a wave climb by quite a modest factor, the economics of this gain could turn out very welcome. Yet statistical significance of the corresponding level might not be detectable by "the naked eye": only controls, comparisons of like with like, and sufficient recorded flights in large enough number would give a clear discrimination. In such a trial, use of directional gyros, and perhaps evaluation of the success or otherwise in holding position by GPS would be advisable.

Obviously these remarks are only a hint of the planning that would be needed for a trial. Results from well planned trials at different sites could be combined; a process that is currently getting a lot of attention in medical statistics.

9. SOME OTHER APPLICATIONS INCLUDING LOW FLYING AND SAFETY PROBLEMS (ALSO APPLICABLE TO POWER FLYING).

(9.i) General Remarks on the Equation of the Curves: Cycloids.

The four bounding curves of the optimal figure-8 belong to a family of curves, the Cycloids, which are well known in other applications. The "simple" cycloid of the textbooks is commonly presented as the path of a point on the rim of a wheel that is rolling along the X-axis. Taking the radius of the wheel as 1, it has the parametric equation:-

$$\overline{\mathbf{OP}} = (\eta - \sin \eta, 1 - \cos \eta)$$

where 0 is the position of the point of contact at the start of motion, and P is that of the moving point when the wheel has turned through the angle η. Comparing equation (8) we see that both equations have coordinates that are linear functions of $\cos \eta$, $\sin \eta$, and η. (In the simplest case above $\sin \eta$ is absent but this is due to the special choice of axes). The point is that we can expect these curves, and others that may arise in further applications, to have properties common to the family.

(9.ii) Final Turns and Low Flying.

Looking at the part of the curve in the lower right-hand quadrant, it can be seen to start with flying across wind, and drifting downwind. Then the turn tightens, and ends directly upwind. The relationship with the crosswind leg of a square circuit, when landing is to be directly into wind, is obvious.

Crashes due to spinning on the final turn, and also in situations such as circling low over a fixed point, for photography or other reasons, have a long history. Too low flying speed is obviously a primary cause. However, there is some effect of low altitude, in upsetting the pilot's perception of rates of turn, and of the approach to a desired heading or ground position.

The pilot's natural inclination is to perceive flying a steady turn as circling, which it is relative to the air, but not relative to the ground. The different pattern of the latter – sometimes very different as has been seen in this study – is only experienced in the comparatively short periods of time spent on final approach and other forms of low flying.

Better understanding of what is happening might reduce the risk of an inappropriate response. A study of the curves involved could help in this, and these curves are the some as, or related to, the figure-8 curves of the present study. This would be applicable to power and as well as glider flying.

(9.iii) Birdwatching.

Just after having obtained the equation of the optimal figure-8, the author was walking in the country and observed a kestrel hovering overhead, facing directly across the road, and hardly moving its wings as there was lift off trees and the hedgerow. Suddenly it decided to change station. It swung across wind, and followed the road in a wide curve which tightened as it approached the upwind direction, when it came to a standstill relative to the ground, once more hovering. The author recognizes that a good deal of what he has put into mathematical form has been known for a very long time.

APPENDIX

Table of values of θ° against windspeed and airspeed.

		W/S											
		0	5	10	15	20	25	30	35	40	45	50	55
A/S	55	180	165	151	139	127	115	103	90	77	62	43	0
	50	180	163	149	135	122	109	95	81	65	45	0	
	45	180	162	146	131	116	101	86	69	48	0		
	40	180	160	142	125	109	92	73	51	0			
	35	180	157	137	118	99	79	54	0				
	30	180	153	131	109	86	59	0					

Remark on the end-values.

The value 180 corresponding to nil wind is consistent with the fact that no turn reversal is necessary: circling will equally well keep the glider near the center. The value 0 is similarly consistent: when A = W the optimal technique is to remain on a heading directly upwind.