The Froude Number as a Predictor of Mountain Lee Wave Phenomenon

Ronald A. Mastaler and Nilton O. Rennó, Ph.D.
Institute of Atmospheric Physics and
Department of Atmospheric Sciences
University of Arizona
Tucson, Arizona, USA

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Summary

We present a physical and mathematical basis to motivate the use of the Froude number as a predictor of the presence and the vertical extent of mountain lee wave. Froude number "profiles" were computed from upwind soundings on days when soaring flights were conducted in the Tehachapi-Owens Valley area of Southern California; the quality of the wave ranged from "poor" to world record setting. For comparison, the Scorer parameter profile was computed for these days. The presence of wind shear is implicitly accounted for in our approach, since shear conditions are necessary to excite various oscillatory modes in the atmosphere. Preliminary results indicate that the Froude number may well differentiate between strong and weak wave, as well as indicate the vertical extent of lift. The Froude number compares favorably with the Scorer parameter, which could not a priori distinguish between strong and weak wave days. More data are needed from wave flights flown under controlled conditions to confirm or refute our hypothesis.

Introduction

The phenomenon of mountain lee waves is well known to many soaring pilots. Flying in wave conditions is characterized by smooth, strong lift to high altitude upwind of the wave crest, with strong sink downwind of the crest. These areas of sink may cause destructive surface winds, and the near surface "rotor" beneath the mountain wave crests is an area of potentially severe turbulence, which can be especially hazardous to light aircraft.

Recently, a world distance record for gliders of nearly 2500 kilometers was set in Argentina, while flying in wave. The world, absolute altitude record for gliders, also conducted in wave is about 15,000 meters (over 49,000 feet). Speed records have been set in wave as well; for example, Jim Payne established a speed record over a 100-km course of nearly 235 km/hr. Clearly, mountain lee wave is a very powerful atmospheric phenomenon, extensive vertically as well as horizontally.

Although surface topography certainly affects the character and intensity of mountain lee wave, we will concentrate on the atmospheric conditions, which promote wave, in regions with favorable topography. As an introduction, we present a basic, intuitive physical explanation of mountain lee wave. We assume two-dimensional flow, perpendicular to a mountain ridge.

Consider an air parcel in a stable airmass moving horizontally (see Fig. 1). As the airmass encounters a mountain ridge, the parcel of air is perturbed upward over the ridge. The pressure of the raised parcel equalizes rapidly with the surrounding air, but the stability of the airmass means that the parcel is now denser than the surrounding air. The parcel slows its ascent, stops and starts to descend. As the parcel is carried horizontally with the zonal wind, its rate of descent increases. The parcel's momentum carries it below its initial, equilibrium (i.e. unperturbed) altitude, where it is now less dense than its surroundings and the parcel is eventually carried upward. Depending on conditions, including the strength of the initial perturbation and the stability profile of the airmass, these "buoyancy oscillations" may undergo many cycles in the lee of the ridge. In addition, if the airmass is humid, lenticular clouds may become visible at the wave crests. These oscillations are analogous to that of a weight hanging vertically on a spring, which when vertically displaced and released oscillate about their equilibrium position, until air friction and internal energy losses in the spring (frictional heating) dampen the oscillations.

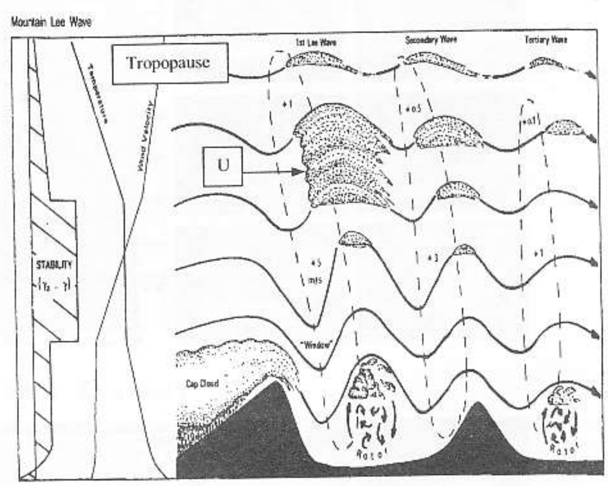

Figure 1. Cross section showing wave patterns in the lee of a mountain from Reichmann [1]. We assume a zonal wind U and tropospheric depth Z_t. The stability (or buoyancy) is obtained from the temperature profile of the troposphere.

According to Reichmann [1], the best conditions for wave are as follows. The airmass must be stable. The wind speed aloft must be greater than about 8 m/s (~15 knots) at ridge level, with nearly constant direction throughout the stable layer. The wind speed should be constant or smoothly increasing with altitude. In addition, the wind direction should be within 30 degrees of normal to the perturbing ridge. These observations have been "quantified" in the dimensional Scorer parameter, which, for wave conditions, should decrease with altitude (Scorer [2], Durran [3], and Reichmann [1]).

The goal of this paper is to quantify the above observations in a nondimensional parameter, whose value indicates both the existence and altitude range of mountain lee wave. This parameter should incorporate measurements of airmass stability and the mean wind speed in the layers of interest. To be useful, this parameter should be easily computable from readily available data (i.e., a daily sounding). As will be shown below, based on scaling of the governing equations and dimensional analysis, such a parameter exists; it is the Froude number, Fr.

To achieve our goal of determining the altitude range of lee wave, we look at modes of atmospheric oscillation, i.e. an infinite number of internal modes (baroclinic modes), as well as an external mode (the barotropic mode). In the barotropic mode, the troposphere moves like a fluid layer of constant density. For lee wave, this is analogous to the standing waves present over, and downstream of, a large submerged obstacle in a river (the water being of constant density). The baroclinic modes in the troposphere result from changes in density in "layers" internal to the troposphere. Such internal waves can be observed in tabletop wave machines, with colored oil and water serving as fluids of similar but different densities. In general, we expect Froude numbers of order unity to indicate the presence of certain modes and lower Froude numbers to indicate the absence of modes. The structure and accompanying vertical wind profile associated with various modes will be discussed in detail below.

As a preliminary test of our hypothesis, we calculated the Froude number using upwind soundings, for the barotropic and first several baroclinic modes, on days when soaring flights were conducted in wave in the Owens Valley, California. These results suggest a correlation between the Froude number of various modes and the presence of lee wave at the altitudes corresponding to these modes. Our hope is that our theoretical treatment of this problem will encourage the soaring community to obtain detailed flight information under controlled conditions, in order to substantiate or disprove our hypothesis.

Above, we have given a simple physical description of mountain lee wave. We will now define the Froude number explicitly and motivate its use in this problem based on dimensional analysis. We will then linearize and scale the governing momentum equations, from which we obtain the Froude number (details of these derivations have been eliminated for brevity). We discuss some of the practical problems in computing the Froude number from soundings. Finally, we present our results and discuss the need for future data collection and research.

The Froude Number

The Buckingham Pi Theorem provides the theoretical basis for using dimensional analysis to describe physical systems. In particular, it states that certain fundamental, physical quantities, characterizing a system may be combined to produce dimensionless products, which provide information on the state of the system. A brief statement of this theorem and its application to this problem are presented in the Appendix, for readers who are unfamiliar with scaling.

The Froude number is defined as the ratio of the inertial force to the gravitational force, i.e.

$$\mathrm{Fr} = (\rho U^2 D^2)/(\rho D^3 \mathrm{g}),$$

where ρ is the fluid density, U is the fluid velocity and D is a length scale (Williams and Elder [4]). The above equation reduces to

$$\mathrm{Fr} = U^2/D\mathrm{g}.$$

Alternatively, the Froude number can be defined as the square root of the above quantity $\mathrm{Fr} = U/\sqrt{D\mathrm{g}}$. This latter expression is the form of the Froude number that we will use.

The length scale D depends on the nature of the system. For "deep water waves", where the fluid depth is large compared to the wavelength, D is simply the wavelength of the waves. For "shallow water waves", where fluid depth is small compared to the wavelength of the waves, D is the fluid depth. In either case, the quantity $\sqrt{D\mathrm{g}}$ is a measure of the phase speed of the waves. Therefore, the Froude number gives the ratio of the flow speed to the phase speed of the waves. We will be considering "shallow water waves", since the wavelength is much greater than the tropospheric depth (see below). In the above definition of Fr, we have assumed gravity waves at a free surface (e.g., an air-water interface). More generally, for waves moving along the interface between any two fluids, each of constant density, g should be replaced by $g' = g(\Delta\rho/\rho)$, where ρ is the density of the lower fluid and Δρ is the density difference between the two fluids. The quantity g' is called reduced or modified gravity. Note that for the interface between air and water that $\Delta\rho/\rho \cong 1$.

We will use various formulations for Fr, but each gives the ratio of the fluid flow speed to the phase speed of the waves. One very important point is that since mountain lee waves are standing waves, the wind speed equals the phase speed of the wave (but in the opposite sense). Therefore, we expect the Froude number to be of order one in the presence of wave conditions.

The Governing Equations

Linearizing the governing equations

We will now reinforce the physical intuition described above, with a rigorous mathematical treatment of the problem.

We start with the equations governing the motion of a parcel in two dimensions (in x and z), presuming that the scale is such that we can ignore Coriolis effects. Although friction is important near the ground, we are primarily interested in the atmosphere above the surface layer and so will take the flow, in our region of interest, to be inviscid. We will also use the Boussinesq approximation, in which density is assumed constant except when density variations result in buoyant forces (Gill [5]), i.e. when gravity provides the restoring force to a parcel. We can justify this approximation, assuming motions in the vertical are small compared to the scale height, so that density differences are small compared to those over the scale height ($\cong$ 8.5 km).

The equations governing the flow in this situation can be written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1a)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g \quad (1b)$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \quad (1c)$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+w\frac{\partial \theta}{\partial z}=0, \quad (1d)$$

where u and w are the wind components in x and z. The potential temperature, θ, is defined as

$$\theta = T\left(\frac{p_s}{p}\right)^{\frac{R}{C_p}} = \frac{p}{\rho R}\left(\frac{p_s}{p}\right)^{\frac{R}{C_p}}, \quad (2)$$

where c_p is the specific heat for dry air at constant pressure, R is the gas constant for dry air, p_s is a reference pressure (usually 1000mb).

We can linearize these equations about a basic state in hydrostatic equilibrium, again assuming vertical accelerations are small compared to gravity. The variables in Eqs. (1) can be decomposed into their mean (with zero subscripts) and perturbation (primed) components:

$$\begin{aligned} \rho &= \rho_0+\rho' & u &= u_0+u' \\ p &= p_0(z)+p' & u_0 &= \text{constant (in } x,t) \quad (3) \\ \theta &= \theta_0(z)+\theta' & w &= w' \end{aligned}$$

We need to relate density (and density perturbations) to measurable quantities, namely the potential temperature and its perturbations. Therefore, from Eq. (2) in the basic state, we have

$$\ln\theta_0=\gamma^{-1}\ln p_0-\ln\rho_0+const. \quad (4)$$

where $\gamma = c_p / c_v$. Detailed derivations of Eq. (4) and the linearized form of the governing Eqs. (1) have been eliminated for brevity. The final form of the linearized equations is as follows:

$$\frac{du'}{dt}=\frac{\partial u'}{\partial t}+u_0\frac{\partial u'}{\partial x}+w'\frac{\partial u_0}{\partial z}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad (5a)$$

$$\frac{dw'}{dt}=\frac{\theta'}{\theta_0}g-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} \quad (5b)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0 \quad (5c)$$

$$\frac{d\theta'}{dt}+w'\frac{\partial \theta_0}{\partial z}=0. \quad (5d)$$

By operating on Eqs. (5a-5c), we can obtain a single equation in terms of w' alone,

$$\frac{d^2}{dt^2}\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2\frac{\partial^2 w'}{\partial x^2}=0, \quad (6)$$

where $N=\sqrt{g/\theta_0 \cdot \partial\theta_0/\partial z}$ is the Brunt-Väisälä frequency.

Scaling the governing equations

The variables in Eq. (6) can be written as follows

$$w'=w=Ww^* \qquad z=Hz^* \qquad x=Lx^*$$

$$t=\frac{L}{U}t^* \quad \text{with} \quad u\cong u_0=Uu^*,$$

where the capital letters indicate characteristic orders of magnitude (W – vertical wind scale, L – horizontal length scale, H – vertical length scale, U – horizontal wind scale, and L/U – advective time scale) and the starred terms are of order one. Substituting these expressions into Eq. (6), we have

$$\frac{d^2}{d((L/U)t^*)^2}\left(\frac{\partial^2(Ww^*)}{\partial(Lx^*)^2}+\frac{\partial^2(Ww^*)}{\partial(Hz^*)^2}\right)+ N^2\frac{\partial^2(Ww^*)}{\partial(Lx^*)^2}=0, \quad (6a)$$

or simplifying Eq. (6a),

$$\frac{U^2}{L^2}\frac{d^2}{dt^{*2}}\left(\frac{W}{L^2}\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{W}{H^2}\frac{\partial^2 w^*}{\partial z^{*2}}\right)= =-N^2\frac{W}{L^2}\frac{\partial^2 w^*}{\partial x^{*2}}. \quad (6b)$$

We now multiply both sides by $L^2/(WN^2)$, making the right hand side of Eq. (6b) of order one,

$$\mathrm{Fr}^2\frac{d^2}{dt^{*2}}\left(\frac{H^2}{L^2}\frac{\partial^2 w^*}{\partial x^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right)=-\frac{\partial^2 w^*}{\partial x^{*2}}. \quad (6c)$$

Since, the horizontal scale L is much larger than the vertical scale H, i.e. $L>>H$, we have $H^2/L^2<<1$, and Eq. (6c) reduces to

$$\mathrm{Fr}^2 \frac{d^2}{dt^{*2}} \frac{\partial^2 w^*}{\partial z^{*2}} = -\frac{\partial^2 w^*}{\partial x^{*2}} \approx 1. \qquad (6d)$$

where again starred terms are of order one and Fr = U/NH is the Froude number for internal waves (Gill [5]). This result tells us that for real N, indicating stable conditions, standing waves will occur, if the flow speed, U, equals the phase speed, c, of the waves, though in the opposite sense. Here NH is proportional to the phase speed (a detailed explanation of the phase speed, c, for the various modes, is given later). Therefore, the Froude number, which is the ratio of these speeds, will be of order one for standing waves. Moreover, since standing waves remain stationary over the perturbation which produces them, energy is constantly being put into the wave. Consequently, the amplitude of the standing wave is greater than that of waves carried downstream, $U>c$, (or upstream, $U<c$) of the perturbation.

Scaling equations containing the Scorer parameter

Durran [3] gives a rigorous dynamical description of mountain lee wave. His basic assumptions are the same as ours, namely, two-dimensional flow of an inviscid, Boussinesq fluid with a large Rossby number, such that Coriolis effects can be ignored (Durran [3]). In addition, Durran assumes steady state, i.e. $\partial/\partial t = 0$, and defines the lower boundary condition to be an infinite set of periodic ridges. Whereas Durran is concerned with waves forced by certain forms of topography (e.g., sinusoidal or bell-shaped ridges; Durran [3]), we are concentrating on the atmospheric conditions that generate standing waves.

Starting with Durran's linearized Eqs. (20.1) – (20.4),

$$u_0 \frac{\partial u'}{\partial x} + w' \frac{\partial u_0}{\partial z} + \frac{\partial P}{\partial x} = 0, \qquad (20.1)$$

$$u_0 \frac{\partial w'}{\partial x} + \frac{\partial P}{\partial z} = b, \qquad (20.2)$$

$$u_0 \frac{\partial b}{\partial x} + N^2 w' = 0, \qquad (20.3)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \qquad (20.4)$$

where $b = g\,\theta'/\theta_0$ is the buoyancy, u_0 is the basic state wind, $P = c_p \theta_0 \pi$, θ_0 is a reference potential temperature, and $\pi = (p/p_0)^{R/c_p}$ is the perturbation Exner function (Durran [3]). All other terms are as defined previously.

We can combine Eqs. (20.1) – (20.4) into a single equation for the vertical perturbation velocity (pg. 473, Durran [3]), w',

$$\left(\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} \right) + l^2 w' = 0, \qquad (7)$$

where

$$l^2 = \frac{N^2}{u_0^{\;2}} - \frac{1}{u_0} \frac{d^2 u_0}{dz^2} \qquad (8)$$

and l is the Scorer parameter (Durran [3]; Reichmann [1]).

As above, we can scale Eq. (7e), substituting the Scorer parameter from Eq. (8),

$$\frac{\partial^2 (Ww^*)}{\partial (Lx^*)^2} + \frac{\partial^2 (Ww^*)}{\partial (Hz^*)^2} + \left(\frac{N^2}{(Uu^*)^2} - \frac{1}{Uu^*} \frac{d^2 Uu^*}{d(Hz^*)^2} \right) \cdot Ww^* = 0. \qquad (9a)$$

By regrouping and multiplying both sides by H^2/W, we obtain

$$\frac{H^2}{L^2} \frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} + \left(\frac{N^2 H^2}{U^2} \frac{1}{u^{*2}} - \frac{1}{u^*} \frac{d^2 u^*}{dz^{*2}} \right) \cdot w^* = 0. \qquad (9b)$$

Again since $H^2/L^2 << 1$, Eq. (9b) reduces to

$$\frac{\partial^2 w^*}{\partial z^{*2}} + \left(\frac{1}{\mathrm{Fr}^2} \frac{1}{u^{*2}} - \frac{1}{u^*} \frac{d^2 u^*}{dz^{*2}} \right) \cdot w^* = 0, \qquad (9c)$$

or

$$\frac{1}{\mathrm{Fr}^2} \frac{w^*}{u^{*2}} = -\frac{\partial^2 w^*}{\partial z^{*2}} + \frac{w^*}{u^*} \frac{d^2 u^*}{dz^{*2}} \approx 1. \qquad (9d)$$

The starred terms are of order one, and Fr = U/NH is the Froude number for internal waves (Gill [5]). In this case, the steady-state assumption essentially forces a solution where the zonal flow speed, U, equals the wave's phase speed, $c \sim NH$ (see below for details), for real-valued N. In other words, this assumption constrains the solution, under stable conditions, to standing waves and a Froude number, $\mathrm{Fr} \cong 1$.

Computation of the Froude Number

Vertical wind structure for the barotropic and baroclinic modes

Before describing how we compute the Froude number for the various modes, we will first describe the barotropic and baroclinic modes, in particular, their vertical profiles. Refer to Fig. 2 in the following descriptions. Note especially, that wind shear is necessary to excite various modes of oscillation. The barotropic mode of oscillation implies that the troposphere moves vertically like a fluid layer of constant density. If the barotropic mode is excited, then the vertical velocity under, or somewhat upwind of, the crest[1] increases steadily from zero at the surface to a maximum at the top of the fluid (the tropopause). The vertical velocity may actually continue to increase into the stratosphere.

Some of our thoughts have been motivated by accounts of flights conducted during the *Jet Stream Project*. The following excerpt from *Exploring the Monster* (Whelan [7]), a history of the *Sierra Wave* and *Jet Stream Projects*, describes wave conditions on March 29, 1955:

> "... the Bishop wave on the 29th was smooth from bottom to top. It also had a tremendously long wavelength, some twenty miles. The first updraft was fully ten miles downwind from the Sierra, almost atop the Whites. Although the wave's lift was initially encountered only 4,400 feet above the airport, its maximum lifting strengths were not encountered until above the tropopause in the stratosphere."

This seems to have been a "barotropic wave" day. Note also, the long wavelength supports our contention that $L >> H$ (see Section 3). Whelan [7] also provides this account on April 1, 1955:

> "Most surprisingly, in addition to the unstable [sic] troposphere being able to "muscle" the stable stratosphere into wave motion (as had first been shown possible by Edgar and Klieforth's flight four years earlier), the stratospheric wave motions were quite different than those below, almost as might occur between adjacent layers of two entirely different fluids."

Presumably the "unstable troposphere", in the above description, means unstable relative to the stratosphere, though still stably-stratified.

Figure 2 shows schematically that each baroclinic mode has a fixed vertical wind speed profile; the extrema are always at the same depth in the fluid. Keep in mind that although we consider each mode separately, the final solution to the vertical wind profile is a superposition of various modes (the barotropic and an infinite number of baroclinic modes). For the first baroclinic mode, the vertical wind profile is a half-period sinusoidal curve with the maximum upward vertical velocity in the middle troposphere. The second baroclinic mode adds a half-sinusoidal period, with the maximum upward velocity at one-fourth the tropopause altitude (and maximum downward velocity at three-fourths the tropopause altitude). The third baroclinic mode adds another half-sinusoidal period with upward vertical velocity maxima at one-sixth and five-sixths the tropopause altitude, and so on (see Fig. 2). When we speak of wave lift, associated with a particular baroclinic mode, we are concerned with the "first maximum" of the vertical velocity profile above the surface (see Fig. 2), since this is where lift is first encountered.

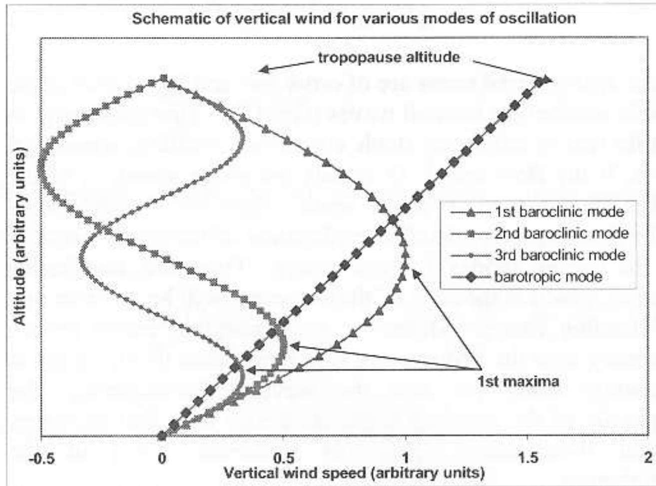

Figure 2. Schematic of the vertical wind structure for the barotropic and first several baroclinic modes. The magnitudeof the wind is arbitrary. The barotropic mode has the highest magnitude vertical wind, with the wind of each subsequent baroclinic mode of lesser magnitude. However, the relative amplitude is not known. When considering the existence of wave, we focus on the altitude of the "first maximum" in the upward vertical wind profile for each mode

The barotropic mode, if present, provides the largest vertical velocity at high altitude, with a maximum at the top of the troposphere. It becomes clear why, if we think of the momentum of the entire troposphere moving en masse. In the first baroclinic mode, a much thinner layer of fluid is moving upward, therefore there is much less momentum transfer. Each subsequent baroclinic mode (see Fig. 2) has a smaller vertical wind magnitude.

The Froude number for the barotropic mode

Recall that for the barotropic mode of oscillation, the troposphere moves like a fluid layer of constant density. Our approach for the barotropic mode of the troposphere is to treat the stratosphere and troposphere as the upper and lower layers of a two-layer system. The Froude number is then $\mathrm{Fr} = U / \sqrt{g(\Delta\rho / \rho) \cdot Z_t}$, where ρ is the mean density of the troposphere and Z_t is the tropospheric depth. For a straightforward two-layer system, $\Delta\rho$ is the density difference between the two fluids, each of constant but different densities. However, we argue that since $\sqrt{g(\Delta\rho / \rho) \cdot Z_t}$ is the phase speed of the wave at the tropopause, the relevant density difference, $\Delta\rho$, is that between the stratosphere and troposphere in the region near the interface. This is similar to the approach Smith and Grubisic [6] used to compute the phase speed of internal gravity waves. However, they used potential temperature differences as a proxy for density differences, across a "well-defined jump in potential temperature" (Smith and Grubisic [6]).

Our problem, therefore, is to determine the mean density of the troposphere and the density difference from our sounding

[1] Here we have ignored the fact that, in general, the wave tilts upwind with altitude.

data. Using temperature soundings, we first compute a mean tropospheric temperature, from which we determine a mean pressure scale height H_p

$$H_p = R\overline{T}/g, \tag{10}$$

where R is the specific gas constant for dry air, $\overline{T}$ is the mean tropospheric temperature (in Kelvin) and g is gravity. The surface density, ρ_0, can be derived from the sounding using the ideal gas law

$$\rho_0 = p_0/RT, \tag{11}$$

where p_0 is the surface pressure and T is the surface temperature. The density profile is then

$$\rho \cong \rho_0 \exp(-z/H_p). \tag{12}$$

The mean density for the troposphere, $\overline{\rho}_t$, is given by

$$\overline{\rho}_t = \frac{\int_0^{Z_t} \rho_0 \exp(-z/H_p)dz}{\int_0^{Z_t} dz} =$$

$$= -\frac{\rho_0 H_p}{Z_t}\left[\exp(-Z_t/H_p)-1\right], \tag{12a}$$

where H_p is the pressure scale height and Z_t the height of the tropopause (specified in the sounding).

To compute the density difference across the tropopause, we compute mean densities for the top one-kilometer layer of the troposphere,

$$\overline{\rho}_{t(top)} = \frac{\int_{Z_t-1km}^{Z_t} \rho_0 \exp(-z/H_p)dz}{\int_{Z_t-1km}^{Z_t} dz} =$$

$$= -\frac{\rho_0 H_p}{1km}\left[\exp\left(-\frac{Z_t}{H_p}\right) - \exp\left(-\frac{(Z_t - 1km)}{H_p}\right)\right] \tag{12b}$$

and the bottom one-kilometer layer of the stratosphere,

$$\overline{\rho}_{s(bottom)} = \frac{\int_{Z_t}^{Z_t+1km} \rho_0 \exp(-z/H_p)dz}{\int_{Z_t}^{Z_t+1km} dz} =$$

$$= -\frac{\rho_0 H_p}{1km}\left[\exp\left(-\frac{(Z_t + 1km)}{H_p}\right) - \exp\left(-\frac{Z_t}{H_p}\right)\right]. \tag{12c}$$

The one-kilometer value for the layer thickness was chosen, based on the assumption that the maximum peak-to-peak amplitude of the wave does not exceed about two kilometers. That is, an air parcel is perturbed no more than about one kilometer (up or down) from its equilibrium level. The Froude number is then

$$\mathrm{Fr} = U / \sqrt{g((\overline{\rho}_{t(top)} - \overline{\rho}_{s(bottom)})/\overline{\rho}_t)\cdot H} \tag{13}$$

where $\overline{\rho}_t$ is the mean density of the troposphere.

With this approach, we use the copious temperature sounding data in the troposphere to obtain an approximate density profile for the atmosphere. Otherwise, due to the lack of temperature sounding data at stratospheric altitudes, we would be forced to choose, rather arbitrarily, a potential temperature difference between the stratosphere and troposphere (as was done by Smith and Grubisic [6]).

The Froude number for the baroclinic modes

The baroclinic modes of oscillation are wave motions internal to a fluid, in our case internal to the troposphere. To obtain the baroclinic modes mathematically, we use the rigid lid approximation, where the vertical displacements of the free surface are considered small compared to internal wave displacements, i.e. the vertical velocity of the tropopause is assumed close to zero. In reality, the motion of the troposphere is a superposition of many modes. Before giving the quite simple form of the Froude number for the baroclinic modes, we will show how this approximation simplifies our treatment of the baroclinic modes and determines their vertical structure.

Equation (6) has wavelike solutions of the form

$$w' = w_0 e^{i(kx+mz-\omega t)}, \tag{14}$$

where k and m are the wavenumbers in the x and z directions, respectively; the real portion of Eq. (14) is the physically meaningful solution. Considering tropospheric waves of large horizontal scale, we can simplify our treatment to waves moving only in the horizontal direction, and Eq. (14) reduces to

$$w' = w_0(z)e^{i(kx-\omega t)}, \tag{14a}$$

where we have performed a separation of variables, to express the solution as a sum of normal modes (Gill [5]). The boundary conditions for the rigid lid approximation specify that the vertical velocity is zero at the surface and the tropopause, Z_t, fixing the vertical structure of the baroclinic modes to a function $w_0(z)$ having sinusoidal form. In the longwave limit, the phase speed of these internal waves is given by

$$c_n^2 \cong N^2 Z_t^2 / n^2\pi^2 \tag{15}$$

where c_n^2 is the phase speed squared, and n=1,2, ... is the baroclinic mode number (Gill [5]).

The Brunt-Väisälä frequency[2], $N^2 = g/\theta_0 \cdot \partial\theta/\partial z$, in the above equation is a measure of stability, analogous to the buoyancy term, $B = g(\Delta\rho/\rho)$, which we used for the barotropic mode. From this equation, we can see that if the potential temperature does not change with altitude, then $N^2 = 0$; this is the criterion for a neutrally stable atmosphere (assuming no condensation). If the potential temperature decreases with altitude, then the atmosphere is unstable and $N^2 < 0$. Conversely, if potential temperature increases with altitude, the atmosphere is stable, $N^2 > 0$ and N is the oscillation frequency of the perturbed parcel[3]. Recall that a stable atmosphere is conducive for lee wave, so N must be real-valued, for gravity waves to be present.

The Froude number, the ratio of fluid flow speed to phase speed, is then

$$\mathrm{Fr}_n = U/c_n = Un\pi/NZ_t \qquad (16)$$

where c_n is the phase speed as defined above and n is the mode number.

To compute the buoyancy frequency N, the $\partial\theta/\partial z$ are calculated for each pair of temperature data points in the sounding, between the surface and tropopause. These values are then averaged to give the mean derivative of potential temperature with height over the troposphere. The quantity θ_0 is the mean potential temperature of the troposphere. From this, we obtain a single, mean buoyancy frequency for the troposphere. The buoyancy frequency should be more properly given by

$$N = \sqrt{\left(\overline{g/\theta_0}\right) \cdot \left(\overline{\partial\theta/\partial z}\right)} \qquad (17)$$

where the overbar indicates the average over the depth of the troposphere. The buoyancy frequency most likely changes over the depth of the troposphere, however, for this first order approximation an average N for the troposphere should suffice.

We now have all the equations needed to compute the Froude numbers for the various modes from the sounding data.

Preliminary Results and Analysis

Brief description of flights

Four wave flights, over the Sierras and Tehachapi Mountains in California, were selected for analysis. These were chosen because we had the most information about these flights, and could obtain upstream sounding information for the days in question. Of particular interest is Robert Harris' world altitude record flight, since it occurred over such a great vertical extent. The wave flights that were analyzed are:

1) February 17, 1986 - Robert Harris' world altitude record, ~15 km (over 49,000 feet).
2) May 6, 2000 - Jim Payne's world record speed flight over a 100-km triangle.
3) May 7, 2000 - a wave flight, described as "good all day 12PM to sunset" (personal correspondence); maximum altitude of ~8 km (about 26,400 feet).
4) June 8, 2000 - described as a poor wave day (personal correspondence), used as a counter example.

The descriptions of the year 2000 flights were sent to us by Jim Payne, and Cindy Brickner of Caracole Soaring (Rosamond, California).

Here is Jim Payne's description of his world record flight (taken from his web site):

> "The second run started at 17:30:21 PDT at 14,598 feet MSL. The first leg of 43.97 km took just over 10 minutes (258 kph). The second leg of 28.5 km into the wind took 9 minutes (190 kph). The last leg of 28.5 km took 6.5 minutes (263 kph). I finished at 11,785 feet MSL. The run around the 100.97 km 28% FAI triangle took 25 minutes and 47 seconds. The GPS shows speed of 234.85 kph (145.93 mph) which betters the World 15-Meter 100 km Triangle Record by 53 kph (29%) and raises my Open Class World Record by 17 kph. It is within a few seconds of the US National 100 km Triangle Speed Record I set in 1996 on a triangle that used the (now retired) rule that allowed the start and finish to be 10 km apart."

The only detailed information on the May 7, 2000 "good wave" flight, is that the maximum altitude was approximately 8000 meters (about 26,400 ft.). We do not know why the pilot terminated his climb at that point (airspace restrictions, fatigue, or the top of the lift?). As for the "poor wave" day (June 8, 2000), we only know that wave was present, but not strong.

For our computations, we used the Oakland sounding, taken at 1200Z (5:00 AM local), as the closest, upstream sounding for these flights. We used morning soundings to test the Froude number as a predictor of wave during the day in question. Wind speed data were missing from the Oakland sounding for February 17, 1986 (1200Z) above about 7,000 meters. We obtained wind speed data at higher altitudes (at the tropopause near 200-mb) using a reanalysis provided by the NOAA-CIRES Climate Diagnostics Center (www.cdc.noaa.gov), Boulder, Colorado. SkewT plots for the three most recent flight days are presented in Fig. 3a; the 200-mb "reanalysis" winds, averaged over February 17, and 18, 1986, are presented in Figure 3b.

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[2] The term buoyancy frequency is generally agreed to be the more physically descriptive. The two terms will be used interchangeably.

[3] As pointed out elsewhere (Gill [5]; Durran [3]), N is actually the highest frequency of a parcel moving purely vertically. In general though, a standing wave tilts upstream with altitude, so the actual oscillation frequency is most likely less than N (which will slightly increase the value of the Froude number).

Analysis of the Scorer parameter

One purpose of this study is to compare the Froude number, as a predictor of wave conditions, with the Scorer parameter, since the Scorer parameter has been used for many years by the soaring community. Therefore, we present the Scorer parameter profile for the troposphere, using the approximation of Durran [3]

$$l^* = N/U. \qquad (18)$$

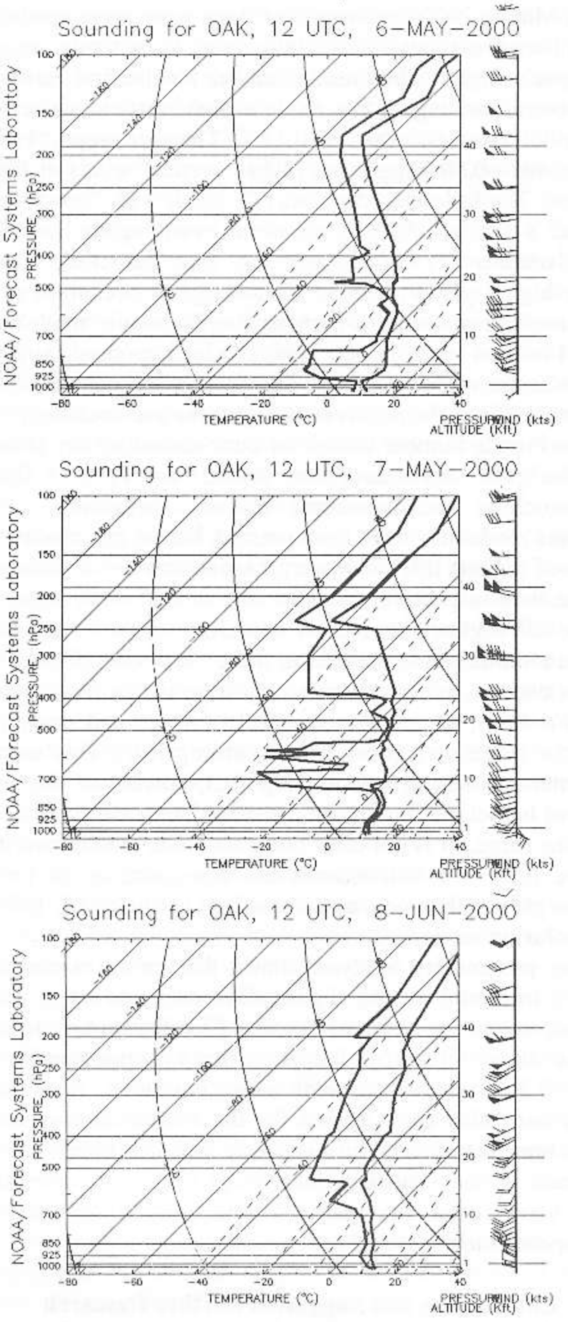

Figure 3a. SkewT plots for three of the wave days analyzed. The upper two plots are for the "good" wave days. The bottom plot is for the "poor" wave day. It should be noted that the wind direction was more consistent for the "good" wave days.

The plot of this profile is given in Fig. 4. The value of N is taken to be the average buoyancy frequency, used to compute the Froude number for the baroclinic modes. The value of U is the zonal wind speed at the indicated altitude.

Let us examine the Scorer parameter profile (Fig. 4). Both Reichmann [1] and Durran [3] indicate that an airmass will have suitable conditions for wave when the Scorer parameter decreases with altitude. Durran [3] is even more explicit and states that: "Waves will be favored whenever the l^* profile decreases significantly with height." The profiles for the good wave soaring days (May 6, and 7, 2000 and February 17, 1986) certainly decrease with altitude. It is not obvious, however, whether their decrease is any more "significant" than on the poor wave soaring day (June 8, 2000). Indeed, the Scorer profile for February 17, 1986, the day of the world altitude record, decreases the least with altitude (about a factor of two over its altitude range versus about a factor of five for the "poor wave" day; see Fig. 4). The criterion that the Scorer parameter decrease significantly with altitude would not have, a priori, differentiated the likelihood of wave on these four days.

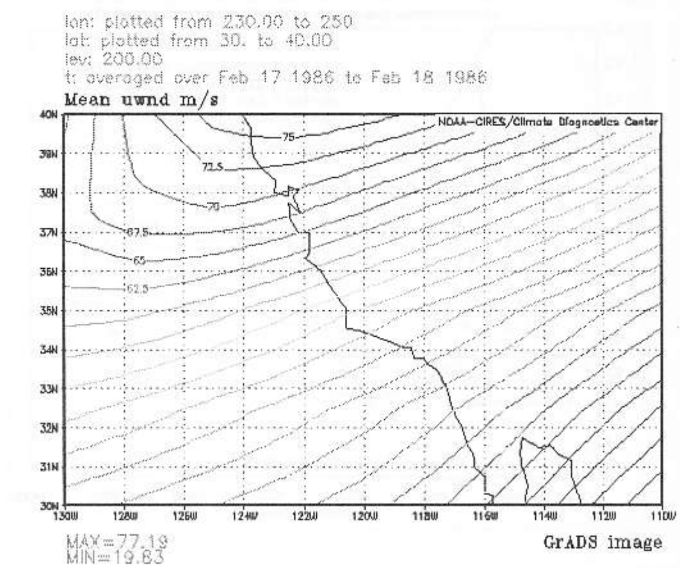

Figure 3b. Winds at 200 mb, the approximate altitude of the tropopause, on the day of Robert Harris' altitude record flight. This reanalysis is provided by the NOAA-CIRES Climate Diagnostics Center, Boulder, Colorado (http://www.cdc.noaa.gov).

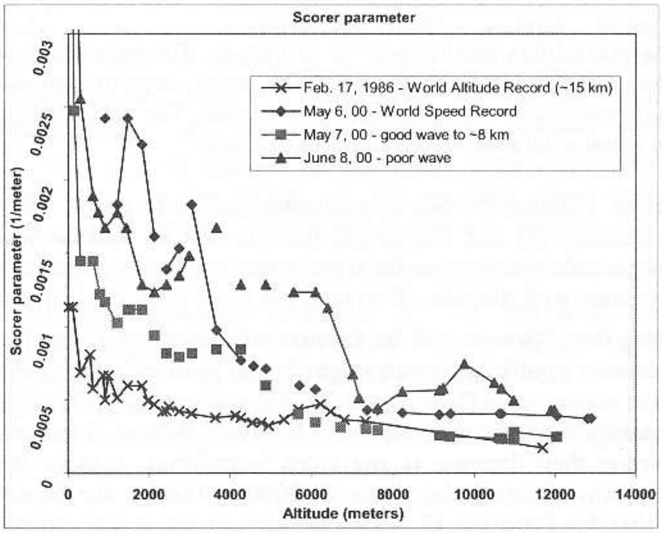

Figure 4. The Scorer parameter for the wave days analyzed; gaps between points indicate missing intermediate data. The criterion that the Scorer parameter decrease with altitude, for wave to be present (Reichmann [1]; Durran [3]), seems to be satisfied for all cases.

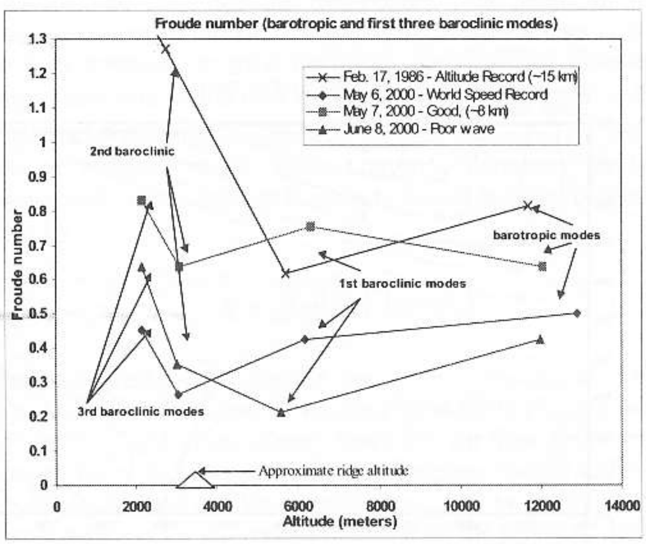

Figure 5. Froude numbers for the wave days analyzed. The highest barotropic Froude number occurs on the day of the record altitude flight. This is consistent with our theory; i.e. a barotropic mode, having increasing climb rate with altitude, would allow the highest altitude flights.

Analysis of the Froude number

To reiterate, our hypothesis predicts that a Froude number of order one for a particular oscillatory mode should indicate the presence of that mode and its associated wind profile. So, for example, if the barotropic mode is excited, the vertical wind speed increases with altitude to the tropopause (or even higher, Whelan [7]); such conditions would be conducive to high altitude flights or speed flights at high altitude. If only the first baroclinic mode is excited then the highest, vertical wind speeds should be located near mid-troposphere; in this case, high-speed flights could be flown near this altitude.

For the Froude number profile plotted in Fig. 5, the rightmost point on each curve indicates the barotropic mode for the troposphere on that day; the next point to the left is the first baroclinic mode, and so on. Referring to Fig. 5, the highest barotropic Froude number (Fr~0.82) is on the day of the world altitude record (February 17, 1986), presumably indicating that this mode was excited. The first baroclinic Froude number is also quite high (Fr~0.62) on this day.

On May 6, 2000, we know that there were wave conditions at least between about 4000-5000 meters, the altitude range of Jim Payne's flight. The Froude number for the first baroclinic mode, corresponding to the mode's peak vertical velocity at about 6000 meters, was Fr~0.43. The barotropic Froude number was ~0.50 suggesting higher vertical winds at higher altitudes. The following day (May 7, 2000, with "good wave" to about 8 km or 26,000') gives an even higher barotropic Froude number (Fr~0.64), with the first baroclinic Froude number higher yet (Fr~0.76). This suggests excitation of the first baroclinic mode (and perhaps the barotropic mode) with high lift at (and above) 6000 meters. Our hypothesis suggests that conditions on May 7, 2000 may well have been stronger at the altitude of Jim Payne's record flight (the previous day).

The Froude number values for our contrasting day (June 8, 2000, the "poor" wave day) were Fr~0.21 and Fr~0.43, for the first baroclinic and barotropic modes, respectively. The barotropic mode may have been present, but no one reached the altitude of highest lift. However, these values are considerably less than their respective values on the "strong" wave days.

For all flights, the second baroclinic modes are located near the altitude of the perturbing ridge. If a second baroclinic mode is excited, as for example may have occurred on May 7, 2000 (Fr~0.64), the combined effect of wave and orographic lift at the ridge may produce very strong lift. This may be experienced as strong smooth ridge lift, upwind of the ridge. The third baroclinic modes, on the other hand, are located well below the ridge. It is not clear how this mode would contribute to wave lift. We will discuss the consequences of Froude numbers greater than one, as occurred on February 17, 1986, in the conclusion section.

This preliminary analysis shows that strong wave, even resulting in record setting flights, does occur on days, and at altitudes, known to have exhibited a Froude number of order one (say, above ~0.5-0.6). It appears that a Froude number less than Fr~0.3 corresponds to weak wave conditions. It should be kept in mind that these values for the Froude number might well be underestimated, since we have assumed purely vertical movement of the restoring (buoyancy) force. As mentioned above, waves often tilt upstream with altitude, therefore the phase speed is lowered and the Froude number higher.

Conclusions and Suggested Further Research

We have presented a theoretical basis for using the Froude number to determine the vertical extent of mountain lee wave. According to our theoretical treatment of this problem, a Froude number of order one for a particular oscillatory mode should indicate wave conditions (lift) in the altitude range of the first maximum of that mode. The Froude number contains measurements of atmospheric stability and zonal wind speed, both important factors for the existence of mountain lee wave. In addition, for a standing wave the Froude number, which is the ratio of the zonal wind speed to the phase speed of the wave, should be of order one.

Our results suggest a connection between the presence of wave and a Froude number of order one ($\mathrm{Fr} \geq\sim 0.5$). A Froude much less than one ($\mathrm{Fr} \leq\sim 0.3$) indicates weak wave or its absence. The Froude number profile also indicates the altitude range of wave lift. The limiting factor, in this study, is the small sampling of data and the lack of flight information regarding the vertical rate of climb as a function of altitude.

Clearly, more data must be taken in wave flights under controlled conditions. The additional data required are the climb rates as a function of altitude. This data would allow us to correlate the maximum vertical wind with a particular oscillatory mode and its associated Froude number. This data would be most useful, if obtained over the entire altitude range of the wave. For this to occur, we need the cooperation of Air Traffic Control to allow flights into Class A airspace (above 18,000 feet). Again, Caracole Soaring seems to have a good relationship with the local ATC center, so many flights above 18,000 feet in the Tehachapi-Owens Valley area may be possible. With a sufficient number of such flights, we should be able to confirm or refute a correlation between the Froude number values and the altitude range of wave.

Finally, we would like to point out that at lower altitudes we computed baroclinic Froude numbers greater than one. These are considered "supercritical" Froude numbers. In the lee of a perturbing ridge, such flows (and even "critical" flows with Fr~1.0) may experience a so-called hydraulic drop, i.e. the wave breaks. Such conditions, at low altitudes, may cause damaging, high winds and surface dust storms in the lee of the ridge. An important study might be to compute the Froude number for these lower altitude modes to correlate their value with the occurrence of dust storms.

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We would like to thank Cindy Brickner of Caracole Soaring for her help. We look forward to working with her in the future, as we test the theory presented here.

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Appendix

The Buckingham Pi Theorem.

The Buckingham pi theorem can be stated as follows (Williams and Elder [4]):

> "Given n parameters (such as length, speed, density, viscosity, force, etc.) and these parameters are composed of a set of m fundamental quantities (... mass, length, and time), then it is possible to express the relation between the parameters in terms of (n minus m) dimensionless products, formed from any n parameters regarded as primary."

What are the n physical parameters, which describe our physical system? We assume an inviscid fluid, that is we ignore viscosity above the surface layer, since we are not interested in how the wave decays downstream from the mountain (i.e. the secondary and tertiary waves). We also assume flow on a scale such that effects of the Earth's rotation can be ignored. Referring to Fig. 1, we can see that the zonal flow U is certainly important, since without it there is no upward perturbation. The buoyancy $B = g(\Delta\rho/\rho)$, at the top of the troposphere, is also important, since it determines the subsequent motion of the perturbed air. Here $\Delta\rho$ is a measure of the density difference between the troposphere and stratosphere. The depth of the fluid Z_t (which we take as the depth of the troposphere) tells us something about the momentum of the system; greater momentum is associated with a deeper fluid layer. Likewise, the mountain height h affects the degree of the upward perturbation. We have five physical parameters (g, U, ρ, Z_t and h), with three fundamental quantities (length, time, and mass). According to the Buckingham Pi theorem there should be two dimensionless quantities, formed from our five physical parameters, which relate various characteristics of the system in a meaningful way.

A first obvious choice is the ratio of the mountain height to the fluid depth (h/Z_t). This is a valid dimensionless number, but since we are interested in the atmospheric conditions leading to wave, we will not be concerned with relations involving h. We can combine the remaining parameters (U and Z_t, along with g and ρ, in the form of buoyancy B) into the ratio of the wind speed to the phase speed. This ratio, $\mathrm{Fr} = U/\sqrt{g(\Delta\rho/\rho)\cdot Z_t}$, is the Froude number (note that this is for the barotropic mode of oscillation).

Alternatively, instead of buoyancy B, we could use a measure of stability, such as the Brunt-Väisälä frequency, $N^2 = g/\theta_0 \cdot \partial\theta_0/\partial z$, where θ is potential temperature, without changing the results of the Buckingham Pi theorem. We still have five physical parameters (g, U, θ, Z_t and h) and three fundamental quantities (length, time and temperature, in the form of potential temperature). In this case, we have $\mathrm{Fr} = U/NZ_t$ where NZ_t is the wave's phase speed. This is the form of the Froude number used by Gill [5] for internal waves.

We will use density differences between the troposphere and stratosphere to compute the Froude number for the barotropic mode ($\mathrm{Fr} = U/\sqrt{g(\Delta\rho/\rho)\cdot Z_t}$). However, we will use the Brunt-Väisälä frequency, $N^2 = g/\theta_0 \cdot \partial\theta_0/\partial z$, to compute the Froude number for the baroclinic modes. The details of these computations will be discussed in the main text.