OPTIMAL DOLPHIN SOARING AS A VARIATION PROBLEM

RISTO ARHO

Department of Mechanics
Helsinki University of Technology
Finland

NOTATION

- c vertical velocity of the air
- v sailplane's airspeed
- w sailplane's vertical velocity with respect to ambient air
- w_{gr} sailplane's vertical velocity with respect to ground

INTRODUCTION

The flight technique that often yields the best result in cross-country flying is the so-called "dolphin motion" (i.e. pulling up in lift, diving through down, no thermal circling). This technique is especially effective in cloudstreet flying, enabling long distances of the course to be covered in straight flight (the dolphin technique is a mode of dynamic soaring).

In this paper we attack the problem of minimum flight time in dolphin soaring. We assume that there is no wind and that the thermal conditions are given along the course to be flown. We further assume that the sailplane's polar is known and that the sailplane is flown in such a way that the polar equation is satisfied all the time, i.e. no transients are supposed to occur. Thus the sailplane goes continuously along the polar from one stationary flight condition to another. This, of course, requires delicate maneuvering.

We solve the minimum flight time problem by means of the calculus of variations. We also apply the results obtained in the numerical derivation of the optimal airspeed policies of Nimbus-2 on the flight courses of two simple examples.

SOLUTION OF THE VARIATIONAL PROBLEM

In the minimum time problem, we are to find a continuously differentiable function $y(x)$ (velocity function) for which the integral

$$I = \int_{x_1}^{x_2} F(y)\, dx \qquad (1)$$

takes on a minimum value and for which the constraint condition

$$J = \int_{x_1}^{x_2} G(x,y)dx = k \qquad (2)$$

where k is a prescribed constant, is satisfied.

Consequently, to solve our variational problem we solve the Euler-equation

$$\frac{\partial}{\partial y}(F + \lambda G) = 0 \qquad (3)$$

where λ is a constant. We carry λ through the calculation and determine it, so that the constraint $\int_{x_1}^{x_2} G\,dx = k$ is satisfied.

OPTIMAL DOLPHIN SOARING

It is customary in graphs, **representing sailplanes' polars to** consider the sinking rate w positive. In this paper, however, we define sinking as negative in order to avoid confusion. Thus motion and transition upwards are positive and downwards negative. If this agreement is kept in mind, we need not resort to vector notation.

Let the horizontal distribution of the vertical velocity component of the air be given by $c(x)$, where x is horizontal distance from a reference point (horizontal velocity component of the air is zero, i.e. no wind). Then the sailplane's vertical velocity **component with respect to ground** is

$$w_{gr} = w + c(x) \qquad (4)$$

where w is the sailplane's vertical velocity with respect to ambient air. Since the sailplane's horizontal ve**locity with respect to ground in ab**sence of wind is approximately (a good approximation) equal to its airspeed, we can write

$$dt = \frac{dx}{\frac{dx}{dt}} \approx \frac{dx}{v} \qquad (5)$$

From Eq. (5) we obtain

$$t = \int dt \approx \int_{x_1}^{x_2} \frac{dx}{v} \qquad (6)$$

In order to minimize the flight time from the point $x = x_1$ to the point $x = x_2$ we have to minimize the integral of Eq. (6). On the other hand, it is often desirable to maintain a constant altitude in the sense that $h(x_1) = h(x_2)$ (between the end points the altitude generally varies), or perhaps it is possible to allow a small altitude difference between the end points. This gives us the constraint:

$$\int_{x_1}^{x_2} w_{gr}\,dt = \Delta h \qquad (7)$$

which by Eqs. (4) and (5) can be written in the form

$$\int_{x_1}^{x_2} [w + c(x)]\,\frac{dx}{v} = \Delta h \qquad (8)$$

Consequently the problem is as follows

$$\min \int_{x_1}^{x_2} \frac{dx}{v}, \qquad (9)$$

$$\text{when} \int_{x_1}^{x_2} [w + c(x)]\,\frac{dx}{v} = \Delta h$$

The problem is clearly the variational problem of Section 3 (the polar equation gives $w = w(v)$). The solution of the problem is $v^*(x)$, which minimizes the flight time and satisfies the altitude constraint. It is also clear for physical reasons that $v^*(x)$ is the minimizing solution, not the maximizing one. Thus we have obtained the optimal airspeed policy in straight flight between the points $x = x_1$ and $x = x_2$. According to this policy, as we shall see, the pilot has to slow down in lifts and hurry on in descending air. Due to this fact this optimal technique has been called the "dolphin motion".

POLAR EQUATION AND OPTIMAL AIRSPEED POLICY

The points of a sailplane's polar $w = w(v)$ give the (v,w)-combinations that are valid in different stationary gliding states. If the pilot wants to change his sailplane's gliding state, he must take an action, which leads to the change - in other words - he uses the joystick. If there is a substantial change in the gliding state and the action is abrupt, a transient follows between the gliding states when the polar equation is not satisfied. If, on the other hand, the pilot operates slowly enough and continuously, no transients follow or they are so slight that they can be ignored. We assume that steering occurs in this manner so that the polar equation is valid all the way through the course.

The general form of the polar equation is

$$w = Av^3 + Bv + C/v \tag{10}$$

where A, B and C are constants. In the region of the "laminar bucket", the approximation

$$w = Av^3 + Bv \tag{11}$$

is often applied. Further for a symmetrical body we can write

$$w = Av^3 + C/v \tag{12}$$

However, it is possible to approximate the polar equation by other kinds of expressions, too. For example, the polynomial

$$w = Av^2 + Bv + C \tag{13}$$

can successfully be fitted to a sailplane's polar data, as we shall see. An accuracy of $\sim 2\%$ is easily obtained by the approximation of Eq. (13). From this point on we suppose that the polar equation is given by Eq. (13).

By Eq. (13) the problem now reads (see Expression (9))

$$\min \int_{x_1}^{x_2} \frac{dx}{v} , \tag{14}$$

$$\text{when} \int_{x_1}^{x_2} \left[Av + B + \frac{C + c(x)}{v} \right] dx = \Delta h$$

where Δh is prescribed. Consequently our Euler-equation (3) takes the form

$$\frac{\partial}{\partial v} \left\{ \frac{1}{v} + \lambda \left[Av + B + \frac{C + c(x)}{v} \right] \right\} = 0 \tag{15}$$

which has the solution

$$v(x) = \sqrt{\frac{1 + \lambda[C + c(x)]}{\lambda A}} \tag{16}$$

The Lagrange-multiplier λ must be so chosen that the altitude constraint is satisfied. Substitution of Eq. (16) to the constraint leads to an uncomfortable integral. For this reason, we have to apply numerical integration and solve λ by iteration.

CASE OF $C(X) = \hat{c} \sin \frac{\pi x}{a}$

In order to progress from this point, we have to fix the function $c(x)$. In order to show clearly the behaviour of "pulling up in lift and diving through down" we now suppose that

$$c(x) = \hat{c} \sin \varphi , \quad (\hat{c} > 0 , \ \varphi = \pi x/a) \tag{17}$$

Then, of course, Eq. (16) takes the form

$$v(\varphi) = \sqrt{\frac{1 + \lambda[C + \hat{c}\sin\varphi]}{\lambda A}} \qquad (18)$$

According to our agreement, both A and C are negative (see Fig. 1). Furthermore, to keep v finite, we have to assume $\lambda \neq 0$. We also assume that $\hat{c} > |C|$.

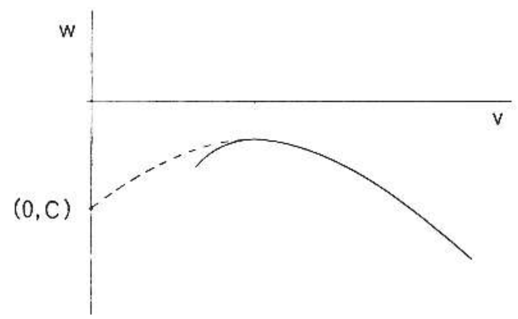

Figure 1. Polar Equation Approximation $w = Av^2 + Bv + C$. Notice the direction of the positive w-axis. Accordingly $A < 0$ and $C < 0$. The line of dashes is the imaginary extension given by the quadratic approximation. The approximation is good in the region of the »laminar buckets«.

Next, suppose that $\lambda > 0$. Then $\lambda A < 0$, since $A < 0$. In order v to be real, we must have $1 + \lambda(C + \hat{c}\sin\varphi) \leq 0$ for all φ. From this we write $\lambda \leq -1/[C + \hat{c}\sin\varphi]$ for all φ that satisfy the condition $C + \hat{c}\sin\varphi > 0$. As $\hat{c} > |C|$, then for example at $\varphi = \pi/2$, $C + \hat{c}\sin\varphi = C + \hat{c} > 0$, and we obtain $\lambda \leq -1/[C + \hat{c}] < 0$, which contradicts our assumption of λ being positive. We thus conclude that

$$\lambda < 0 \qquad (19)$$

Inequality (19) and the fact that $A < 0$ show that $\lambda A > 0$. As v is real, we must have $1 + \lambda[C + \hat{c}\sin\varphi] \geq 0$ for all φ. From this we write $\lambda \geq -1/[C + \hat{c}\sin\varphi]$ for all φ that satisfy the condition $C + \hat{c}\sin\varphi > 0$. Then $\lambda \geq \max\{-1/\{C + \hat{c}\sin\varphi]\}$, when $C + \hat{c}\sin\varphi > 0$, i.e. $\lambda \geq -1/[C + \hat{c}]$. When $C + \hat{c}\sin\varphi \leq 0$, we have $1 + \lambda[C + \hat{c}\sin\varphi] > 0$, and v is real ($\lambda A > 0$). As a summary we write

$$-1/[C + \hat{c}] \leq \lambda < 0 \quad (\hat{c} > |C|) \qquad (20)$$

Now our constraint takes the form (see Expression (14))

$$\int_{x_1}^{x_2} \left[Av + B + \frac{C + \hat{c}\sin\frac{\pi x}{a}}{v}\right] dx =$$

$$\frac{a}{\pi}\int_{\varphi_1}^{\varphi_2} \left[Av + B + \frac{C + \hat{c}\sin\varphi}{v}\right] d\varphi = \Delta h \qquad (21)$$

Consequently, the optimal airspeed function is given by Eq. (18) where λ must be so chosen that Eq. (21) is satisfied. In Eq. (21) Δh is a prescribed constant. Since λ must be determined by iteration, it is useful to notice the range of λ given by Expression (20).

TWO EXAMPLES

We now choose Nimbus-2 as the sailplane. From (1) we obtain the polar data for Nimbus-2. According to the manufacturer, the minimum sink rate 0,48 m/s is achieved at 75 km/h. At 160 km/h, the sink rate is 1,52 m/s. If we fit the polynomial of Eq. (13) to the data in such a way that the approximate polar curve has the maximum at 75 km/h, -0,48 m/s and goes through the point 160 km/h, -1,52 m/s we obtain (1)

$$w = Av^2 + Bv + C, \quad v > v_{stall} = 64 \text{ km/h}$$

where

$$A = -0{,}001866 \text{ s/m} \qquad (22a)$$

$$B = 0{,}07775 \qquad (22b)$$

$$C = -1{,}290 \text{ m/s} \qquad (22c)$$

and v is in m/s. From Table 1 we see that the approximation given by Eqs. (13), (22a), (22b), (22c) is quite satisfactory for our purpose.

TABLE 1. POLAR APPROXIMATION OF NIMBUS-2. w_{ac} IS THE ACTUAL SINK RATE AND w_{appr} THE APPROXIMATE SINK RATE.

v(km/h)	$-w_{ac}$(m/s)	$-w_{appr}$(m/s)
75	0,48	0,48
100	0,58	0,57
140	1,11	1,09
160	1,52	1,52
180	2,02	2,07

Let us further choose

$$c = 2 \text{ m/s} \qquad (23)$$

Example 1. We now assume that we fly through a lift and the subsequent down of the curve $c(x) = 2 \sin \frac{\pi x}{a}$ m/s (see Fig. 2). Then in Eq. (21) evidently $\varphi_1 = 0$ and $\varphi_2 = 2\pi$. We choose a = 2 km and Δh = -70 m (i.e., during the flight of 4 km the altitude is decreased by 70 m), and so we write Eq. (21) to the form

$$\int_0^{2\pi} \left[Av + B + \frac{C + \hat{c} \sin \varphi}{v} \right] d\varphi = \frac{\pi \Delta h}{a} \approx -0{,}110 \qquad (24)$$

where v is given by Eq. (18). All the necessary constants A, B etc., have been given numerical values. It is only left to fix λ so that Eq. (24) is valid. From Expression (20) and Eqs. (22c) and (23) we write

$$-1{,}41 \text{ s/m} \leqslant \lambda < 0 \qquad (25)$$

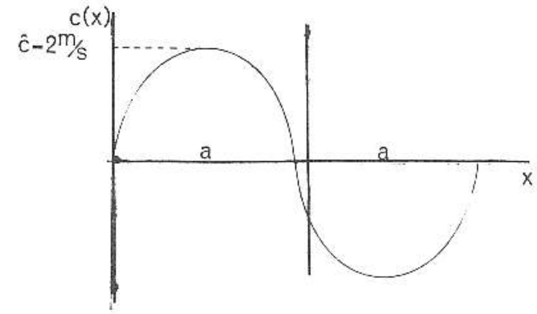

Figure 2. Sinusoidal Variation of c(x).

In this interval $\lambda \approx -0{,}66$ s/m is found to satisfy Eq. (24). This value of λ was accomplished by iteration. In each step a value of λ was chosen and the integral of Eq. (24) was computed by the method of Simpson. The iteration was continued until the integral assumed the value -0,110. With $\lambda = -0{,}66$ s/m Eq. (18) gives (A, C and $\hat{c}$ are given by Eqs. (22a), (22c), and (23) respectively) the airspeed variation $v^*(\varphi)$, which is tabulated in Table 2. The technique of "pulling up in lift and diving through down" is very clearly demonstrated. The minimum airspeed is attained at $\varphi = \pi/2$ and it is 75 km/h, i.e. quite near the stall velocity, which is 64 km/h (7). Thus the altitude decrease could not be significantly smaller than 70 m. when 2a = 4 km. Consequently this example also demonstrates a situation, where the altitude loss is minimal.

TABLE 2. AIRSPEED v* (φ) OF EXAMPLE 1.

φ	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
x(km)	0	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0
v*(km/h)	140	98	75	98	140	171	183	171	140
c(m/s)	0	1,414	2,0	1,414	0	-1,414	-2,0	-1,414	0

Example 2. We next fly our Nimbus-2 through the lift part of c(x) of Figure 2 and choose $\Delta h = 0$. Eq. (21) now reads

$$\int_0^{\pi} \left[Av + B + \frac{C + \hat{c}\sin\varphi}{v} \right] d\varphi = 0 \quad (26)$$

Again v is given by Eq. (18). Now with $\lambda = -0,30$ s/m the integral of Eq. (26) assumes the value -0,00039 so that $\lambda = -0,30$ s/m will do for the approximate solution. Eq. (18) gives the airspeed variation $v^*(\varphi)$, which is tabulated in Table 3. We see that to minimize the flight time through a single lift we are able to fly with quite high speeds, although $\Delta h = 0$.

TABLE 3. AIRSPEED v*(φ) OF EXAMPLE 2.

φ	0	π/4	π/2	3π/4	π
v*(km/h)	179	149	135	149	179
c(m/s)	0	1,414	2,0	1,414	0

SUMMARY

In this paper we have solved the problem of minimum flight time in dynamic soaring by the application of calculus of variations. The sailplane is assumed to go from one stationary flight condition to another continuously so that the polar equation is satisfied all the time along the course. Consequently all possible transients are ignored.

The optimal airspeed policy is derived by applying a quadratic polar equation approximation. The application of the "laminar bucket" approximation of Eq. (11) would lead to a cubic root expression of the optimal airspeed, i.e. no difficulties would have arised, if it had been applied. We, however, apply the quadratic polar equation approximation, because it can be fitted to a sailplane's polar data quite satisfactorily, and because the square root (see Eq. (16)) operation is more easily effected than the cubic root operation in manual numerical calculations.

We treat the special case of sinusoidal lift variation and give two examples, the first of which clearly demonstrates the optimal "dolphin motion."

BIBLIOGRAPHY

Inglesby, J. V., Dynamic soaring. Sailplane-Gliding (1968) Feb.-March, pp. 6-10.

Bohli, H., Optimale Delphinfluggeschwindigkeit auf Streckenflugen. Aero-Revue (1971) 8, pp. 391-393.

Mozdyniewicz, W., Cloudstreet flying. Soaring (1971) 11, pp. 26-27.

Fritsch, E., Zum dynamischen Segelflug. Aero-Revue (1971) 12, pp. 669-672, (1972)1, p.40.

Uotila, J., Tasoituskertoimet purjelentokilpailuissa (Glider handicapping coefficients). Diploma thesis, Helsinki University of Technology, 1972.