CLOUD-STREET FLYING

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NOTATION

C_D	Drag coefficient
C_L	Lift coefficient
D	Drag, assumed to be $D(U)$ in a constant-density atmosphere
$F(U)$	A function such that $T = \int F(U)dx$
F^*	$= F + \lambda G$
$G(U,x)$	A function such that $H_e = \int G(U,x)dx$
g	Acceleration due to gravity
h	True height
h_e	Energy height
H_e	Total change of energy height
L	Lift
t	Time
T	Total flight time
U	Forward speed
U_o	Speed corresponding to $(L/D)_{max}$
$\bar{U}$	$= U/U_o$
U_{cc}	Average cross-country speed
$\bar{U}_{cc}$	$= U_{cc}/U_o$
v_s	Rate of sink
v_{so}	Rate of sink at U_o
$\bar{v}_s$	$= v_s/v_{so}$
v_c	Rate of climb
$\bar{v}_c$	$= v_c/v_{so}$
w	Strength of up-current
$\bar{w}$	$= w/v_{so}$
w^*	$= 1/\lambda$
$\bar{w}^*$	$= w^*/v_{so}$
W	Weight
x	Distance along flight path
λ	A Lagrange multiplier, constant for an optimum flight profile
θ	Flight path slope (positive nose-up)

Suffices are explained in the text.

INTRODUCTION

The criterion for the optimum inter-thermal speed was first published in its simplest form by Barringer in 1940 and has since been elaborated to deal with more realistic situations, notably by MacCready. All of these analyses assume a "normal" cross-country flight in which the sailplane generally gains height by circling in thermals.

Kronfeld's "Austria" was designed in the hope that it would be possible to carry out cross-country flights without circling in the thermals but simply flying straight through them at a low forward speed. At that time, the performance available from even the most refined machines was inadequate for sustained flight in such a fashion and it is only recently that the performance of sailplanes has become so good that significant portions of flights may be carried out without circling. Of course, it has been possible almost since the beginnings of thermal soaring to take advantage of cloud streets, where one finds an almost continuous line of lift or a well-defined closely-spaced succession of thermals.

Whilst the title of this paper refers specifically to cloud streets, its analysis is applicable to any cross-country flight carried out without circling. A criterion for optimising such flights has not been previously proposed, to the best knowledge of the author, doubtless because even a simplified analysis can involve several independent variables. Moreover, as is formally the case in any flight, the forward speed is not necessarily constant and hence the equation of motion involves an acceleration term. Consequently, various integrands similar to those derived in the analysis below are functions (inter alia) of the derivative of forward speed with respect to some other quantity such as time, distance or height, as may be convenient. The analysis then becomes an exercise in the Calculus of Variations, similar to that of the Reference. However, the introduction of the energy height concept serves to eliminate acceleration terms and the problem effectively becomes one in ordinary calculus.

This leads to a result of such a nature that its practical application is difficult. However, it does enable one to assess the conditions under which such flights are possible and indicates the maximum attainable performance.

The analysis below is only concerned with those parts of the flight path having a small inclination to the horizontal. It can also be shown by the methods of the Reference that, if the sailplane's speed at any instant does not correspond with the optimum speed appropriate to the prevailing conditions, it should be adjusted by performing a vertical climb or dive. In reality, such maneuvers are both impracticable and unnecessary but this result does suggest that it is advantageous, when adjusting the speed to the prevailing optimum, to do so as rapidly as is practicable. The analysis neglects the effects of transitions between different conditions of flight, in that it implicitly assumes that the load factor is always unity. Other things being equal, push-over and pull-up maneuvers will produce decreases and increases in induced drag, respectively, so to some extent the effects of a series of such maneuvers will be self-cancelling so far as the overall dissipation of energy is concerned.

ANALYSIS

Consider a sailplane flying on a constant heading. Let x denote distance along the flight path and w (positive upwards) the local vertical velocity of the air. To an external observer, w will in general be a function of both x and time t but, from the point of view of the pilot, it may be regarded as a function of x only. Suppose that the instantaneous forward speed of the sailplane is U. Assume also that the air density is substantially the same as the standard sea-level value.

The time to travel from x_1 to x_2 will be

$$T = \int_{x_1}^{x_2} \frac{dx}{U} \qquad (1)$$

The equation of motion of the sailplane along its flight path will be, in still air,

$$D + W \sin\theta + \frac{W}{g}\frac{dU}{dt} = 0, \qquad (2)$$

where θ is positive nose-up (See Fig.1

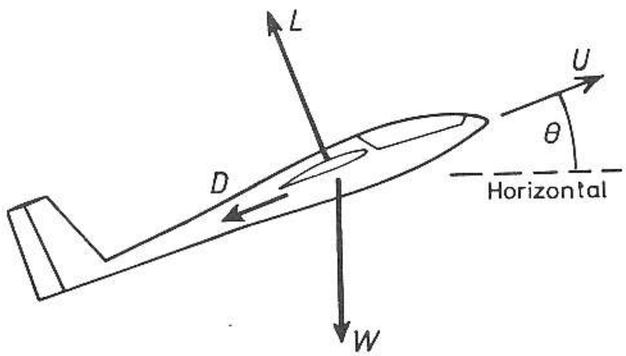

Figure 1

If the energy height is h_e, where

$$h_e = h + U^2/2g \;,$$

and h = true height, then

$$\frac{dh_e}{dt} = \frac{dh}{dt} + \frac{U}{g}\frac{dU}{dt}$$
$$= U \sin\theta + \frac{U}{g}\frac{dU}{dt} \,,$$

and from (2)

$$\frac{dh_e}{dt} = -\frac{DU}{W} \qquad (3)$$

But $\frac{DU}{W}$ is the rate of sink of the sailplane, v_s, when flying steadily at speed U.

In the presence of the upcurrent w, the total rate of change of energy height will be

$$\left[\frac{dh_e}{dt}\right]_{tot} = w - v_s. \qquad (4)$$

The total change of energy height between x_1 and x_2 will be:

$$H_e = \int_{x_1}^{x_2} \left[\frac{dh_e}{dt}\right]_{tot} \frac{dt}{dx}\, dx$$

$$= \int_{x_1}^{x_2} (w - v_s)\, \frac{1}{U}\, dx. \qquad (5)$$

Let us suppose that we wish to fly in such a fashion that, for a given $(x_2 - x_1)$, T is a minimum and $H_e = 0$. This is not the only criterion which could be applied but it represents a simple case analogous to the usual criterion for analysing cross-country flying.

T is of the form $\int F(u)dx$ and

H_e is of the form $\int G(U,x)dx$.

It therefore follows that the criterion to be satisfied is

$$\frac{\partial F^*}{\partial U} = 0,$$

where $$F^* = F + \lambda G$$

$$= \frac{1}{U} + \frac{\lambda}{U}(w - v_s), \qquad (6)$$

and λ is a constant Lagrange multiplier.

So the criterion is:

$$-\frac{1}{U^2} - \frac{\lambda}{U^2}(w - v_s) - \frac{\lambda}{U}\frac{\partial v_s}{\partial U} = 0,$$

since v_s is a function of U only.

This can be re-arranged to give

$$\frac{\partial v_s}{\partial U} = \frac{v_s - w - \frac{1}{\lambda}}{U},$$

or, since $\frac{1}{\lambda}$ must clearly have the dimension of velocity,

$$\frac{\partial v_s}{\partial U} = \frac{v_s - w - w^*}{U}. \qquad (7)$$

The criterion expressed by eqn. (7) is shown graphically in Fig. 2.

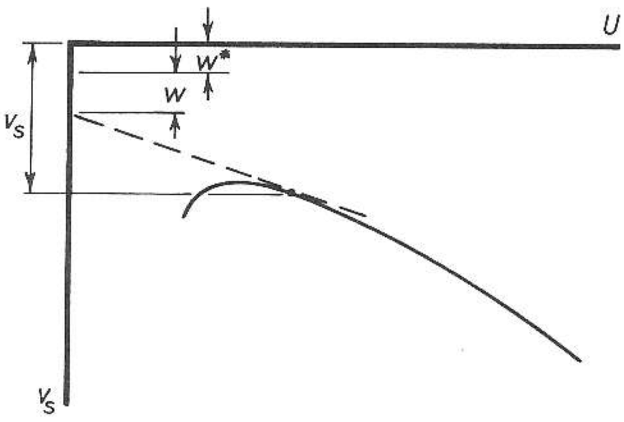

Figure 2

This is the standard "MacCready" situation. (Indeed any optimum flight path for a sailplane gives a similar result, leaving w* to be interpreted according to the circumstances.) w* is, in effect, the zero-setting of the MacCready ring and, whilst the diagram is drawn for a positive value (equivalent to setting the datum opposite some rate of sink figure), the sign of w* remains to be determined. In practice, it will often be negative

It should be noted that circumstances can arise in which one should fly at less than the speed for minimum rate of sink: in other words, it may be advantageous to spend a long time in the upcurrent at the expense of some increase in rate of sink.

From the pilot's point of view, this analysis contains a severe difficulty: w^* is ultimately determined by the condition that $H_e = 0$ and hence requires a knowledge of w as a function of x over the distance $x_2 - x_1$. Unfortunately, the pilot has no powers of prophecy. When flying under a cloud-street, the pilot may initially wish to gain height (on the average) until he is reasonably close to cloud-base and then adjust the MacCready ring by a process of successive approximation so that, overall, there is no net change of height. In real life, there tends to be insufficient time to make the adjustments other than very approximately.

Illustrative calculations for a fixed-geometry sailplane

For a sailplane with a parabolic $C_D - C_L$ curve, the performance polar may be described by

$$\frac{v_s}{v_{so}} = \bar{v}_s = \frac{1}{2}\left(\bar{U}^3 + \frac{1}{\bar{U}}\right), \qquad (8)$$

where $\bar{U} = U/U_o$ and both v_{so} and U_o relate to the $(L/D)_{max}$ condition.

Let $\bar{w} = w/v_{so}$ and $\bar{w}^*/v_{so}$. Then in dimensionless terms, the criterion of eqn. (7) becomes

$$\frac{\partial \bar{v}_s}{\partial \bar{U}} = \frac{\bar{v}_s - \bar{w} - \bar{w}^*}{\bar{U}}. \qquad (9)$$

This is illustrated in Fig. 3. From (8) and (9)

$$\frac{1}{2}\left(3\bar{U} - \frac{1}{\bar{U}^2}\right) = \frac{1}{2}\left(\bar{U}^2 + \frac{1}{\bar{U}^2}\right) \frac{\bar{w} + \bar{w}^*}{\bar{U}}$$

whence $$\bar{w} + \bar{w}^* = \frac{1}{\bar{U}} - \bar{U}^3. \qquad (10)$$

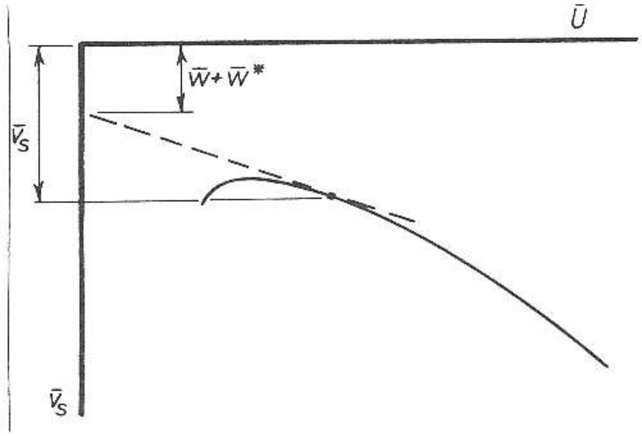

Figure 3

Suppose for the sake of simplicity, that a distance x_1, in which the upwards velocity of the air has the constant value w, is covered at an optimum speed U_1 and a distance x_2, in which $w = 0$, is covered at the corresponding optimum speed U_2.

Then from (10)

$$\bar{w} + \bar{w}^* = \frac{1}{\bar{U}_1} - \bar{U}_1^3,$$

and $$\bar{w}^* = \frac{1}{\bar{U}_2} - \bar{U}_2^3,$$

whence $$\bar{w} = \frac{1}{\bar{U}_1} - \bar{U}_1^3 - \frac{1}{\bar{U}_2} + \bar{U}_2^3 \,. \qquad (11)$$

Also, the rate of climb over distance x_1 will be $v_c = w - v_{s1}$.

For zero overall height change:

$$\frac{x_1 \bar{v}_c}{\bar{U}_1} = \frac{x_2 \bar{v}_{s2}}{\bar{U}_2} \,. \qquad (12)$$

(Strictly, since the previous theory dealt with energy heights rather than true heights, this expression should include a kinetic energy correction.)

So

$$\frac{x_1}{W_2} = \frac{\bar{v}_{s2}}{\bar{U}_2} \cdot \frac{\bar{U}_1}{(w - v_{s1})} \quad (13)$$

Since $\bar{v}_{s1}$ and $\bar{v}_{s2}$ are functions of $\bar{U}_1$ and $\bar{U}_2$ respectively, (11) and (13) can in principle be solved simultaneously to give $\bar{U}_1$ and $\bar{U}_2$ if $\bar{w}$ and x_1/x_2 are known.

It is interesting to consider what combinations of thermal strength ($\bar{w}$, in effect) and distance ratio x_1/x_2 are required to maintain continuous flight. One obvious particular case occurs when $\bar{U}_2/\bar{v}_{s2}$ is a maximum (i.e., when the sailplane is flown at $(L/D)_{max}$ over the distance x_2). This will correspond to $\bar{U}_2 = 1$, $\bar{v}_{s2} = 1$.

Under these conditions

$$\bar{w}^* = 0,$$

$$\bar{w} = \frac{1}{\bar{U}_1} - \bar{U}_1^3 \quad (14)$$

$$\frac{x_1}{x_2} = \frac{2\bar{U}_1^2}{1 - 3\bar{U}_1^4} \quad (15)$$

Eliminating U_1 from (14) and (15) gives a relationship between $\bar{w}$ and the least value of x_1/x_2 which will just permit continuous flight. It is apparent that $\bar{U}_1^4 < 1/3$, from (15). Now, from (8), $\bar{U}_1^4 = 1/3$ corresponds to $\bar{v}_{smin}$ so, as is apparent on physical grounds, the limiting case corresponds to flying at minimum sink in a continuous up-current of the same strength (i.e., $x_1/x_2 = \infty$, $\bar{w} = 2/3^{0.75}$, $\bar{v}_{s1} = 2/3^{0.75}$).

The choice of values of $\bar{U}_1$ is therefore very limited: the maximum value is $3^{-\frac{1}{4}}$ and the minimum value is that corresponding to the stall.

A few results are given in Table 1.

TABLE 1

$\bar{U}_1$	x_1/x_2	w
0.759	∞	0.877
0.75	20.4	0.91
0.70	3.50	1.09
0.65	1.87	1.27

$\bar{w} = \dfrac{\text{Thermal strength}}{\text{Rate of sink of glider at best L/D}}$

$\bar{U}_1 = \dfrac{\text{Speed}}{\text{Speed at best L/D}}$

These results are not realistic because we have imposed the condition that the average speed shall be a max., even very weak thermals require the glider to be flown at unrealistically low speeds. The expression we have used for the performance (8), has no implied lower limit to $\bar{U}$. It would be better to assume that the sailplane is never flown at a speed less than its speed for min. sink, in which case, in examining the limiting conditions for continuous flight, we abandon the maximum average concept. The sailplane is flown at min. sink in the rising air and at its best gliding angle in the still air.

Inserting $\bar{U}_2 = \bar{v}_{s2} = 1$, $\bar{U}_1 = 3^{-\frac{1}{4}}$ and $\bar{v}_{s1} = 2/3^{0.75}$ in (13), this becomes approximately:

$$\frac{x_1}{x_2} = \frac{0.759}{\bar{w} - 0.878} \quad (16)$$

Figures obtained from eqn. (16) are given in Table 2.

TABLE 2

$\bar{w}$	x_1/x_2	$\frac{x_1}{x_1 + x_2}$
0.878	∞	1
1.0	6.22	0.86
1.5	1.22	0.55
2.0	0.675	0.403
3.0	0.375	0.272
4.0	0.243	0.196
5.0	0.184	0.155
6.0	0.148	0.129
7.0	0.124	0.110
8.0	0.1065	0.096

$\bar{w}$ = $\frac{\text{Thermal strength}}{\text{Rate of sink of sailplane at best L/D}}$

x_1 = distance in rising air x_2 = distance in still air

If we now consider in general terms the case of achieving maximum average speed, we can assign some likely constant value to $\bar{U}_1$ and then consider a series of values of $\bar{U}_2$. From (11), we can obtain the value of $\bar{w}$. Since $\bar{v}_{s1}$ and $\bar{v}_{s2}$ are simply related to $\bar{U}_1$ and $\bar{U}_2$ respectively, x_1/x_2 can be found from (13). A more useful quantity is $(x_1 + x_2)/x_1$, i.e., the ratio of the total distance to the distance traversed in lift. It is then possible to derive the non-dimensional average cross-country speed, $\bar{U}_{cc}$, since

$$\bar{U}_{cc} = \frac{\frac{x_1}{x_2}\bar{U}_1 + \bar{U}_2}{1 + x_1/x_2} . \qquad (17)$$

For the present purposes, the assumed values of $\bar{U}_1$ were 0.7 (for the sake of illustration; slightly less than the speed for minimum rate of sink), 0.759 (speed for minimum rate of sink), 1.0 (speed for best gliding angle) and 1.2. Values of $\bar{U}_2$ up to 2.0 were taken.

The results are presented in Table 3 and in Fig. 4 on $(x_1 + x_2)/x_1$, $\bar{w}$, axes. Lines of constant $\bar{U}_1$, $\bar{U}_2$ and $\bar{U}_{cc}$ are drawn.

TABLE 3

	$\bar{U}_2$	$\bar{w}$	$\frac{x_1 + x_2}{x_1}$	$\bar{U}_{cc}$
$\bar{U}_1 = 0.70$, $\bar{v}_{s1} = 0.886$	1.0	1.086	1.287	0.765
	1.1	1.508	1.8-9	0.886
	1.2	1.981	2.469	0.995
	1.4	3.116	3.584	1.207
	1.6	4.557	4.545	1.398
	1.8	6.363	5.405	1.600
	2.0	8.586	6.173	1.785
$\bar{U}_1 = 0.759$, $\bar{v}_{s1} = 0.878$	1.0	0.878	1.0	0.759
	1.1	1.300	1.543	0.878
	1.2	1.773	2.114	38
	1.4	2.908	3.164	1.197
	1.6	4.349	4.115	1.392
	1.8	6.155	4.926	1.590
	2.0	8.378	5.649	1.783
$\bar{U}_1 = 1.0$, $\bar{v}_{s1} = 1.0$	1.22	1.0	1.0	1.0
	1.4	2.030	1.838	1.182
	1.6	3.471	2.710	1.378
	1.8	5.277	3.401	1.564
	2.0	7.500	4.048	1.750
$\bar{U}_1 = 1.2$, $\bar{v}_{s1} = 1.120$	1.395	1.120	1.0	1.2
	1.4	1.135	1.010	1.2
	1.6	2.576	1.825	1.382
	1.8	4.382	2.545	1.565
	2.0	6.605	3.145	1.745

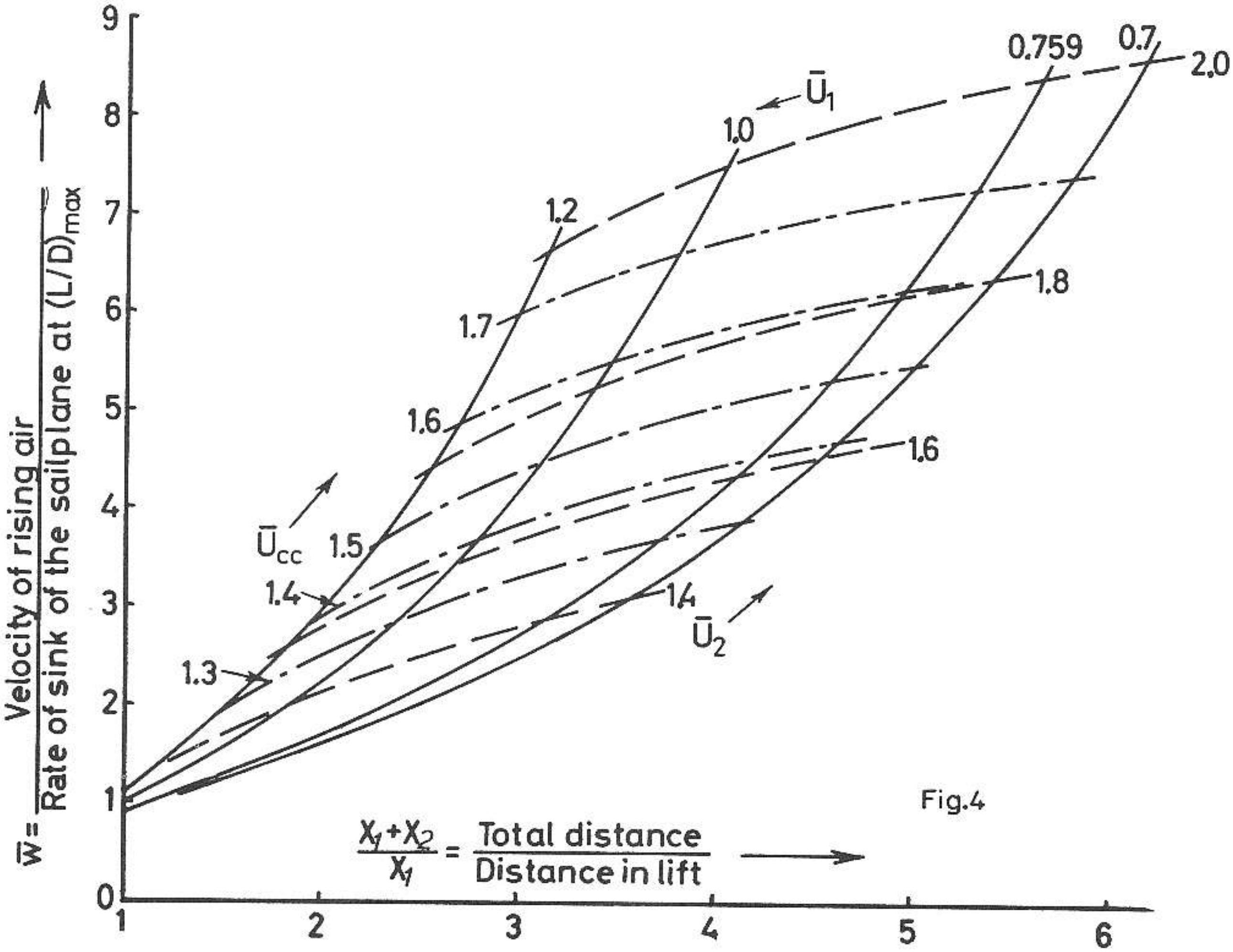

Figure 4

Numerical example

Consider a sailplane whose $(L/D)_{max}$ is 42 at 42 knots EAS.

If $\bar{w}$ = 4 over 25% of the flight path, the upcurrent strength would be 4 knots. Flown at $\bar{U}_1$ = 0.759, the rate of climb would be 4 - 0.878 = 3.122 knots. $\bar{U}_{cc}$ would be about 1.34, corresponding to an average speed of 56.2 knots. The appropriate $\bar{U}_2$ would be 1.55, or 65 knots.

For the same glider with the same vertical air velocity extending over one-third of the distance the results become:

Speed to fly in lift:	42 knots
Rate of climb:	3 knots
Speed to fly between lift:	70 knots
Average speed:	60 knots

CONCLUSIONS

If a cross-country flight is carried out in conditions which permit it to be made without circling, the criterion which must be applied in order to achieve maximum average speed is similar to the MacCready criterion for determining optimum speed between thermals in a normal cross-country flight. However, the datum vertical velocity (denoted by w* in the analysis, and corresponding to the datum setting of a MacCready variometer ring) is determined by the overall distribution of vertical air velocity along the flight path together with some overall condition such as zero change of energy height between the beginning and the end of the flight. In practice, a pilot would have to proceed by a process of successive approximation.

Some calculations have been made for fixed-geometry sailplanes flying through air which has a constant vertical velocity over part of the flight path and is at rest elsewhere.

REFERENCE

Miele, A. and Cappellari, J.O.; Approximate Solutions to Optimum Flight Trajectories for a Turbojet-Powered Aircraft, NASA Tech. Note D-152, Sept. 1959.